bijections from ordinals into a given set

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```
In[1]:= SetDirectory["l:"]; << goedel.11jul07a

:Package Title: goedel.11jul07a                 2011 July 7 at 6:00 p.m.

Loading takes about eleven minutes, half that time due to builtin pauses.

It is now:  2011 Jul 8 at 9:12

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2011 Jul 8 at 9:23
```

summary

In this notebook it is shown that for any set **x**, the class **BIJ ⋂ image[inverse[IMAGE[FIRST]], Ω] ⋂ P[V × x]** of bijections from ordinals into **x** is a set. A key ingredient is the theorem that the Hartogs number of a set is a set, a fact derived 2010 November 8 in the posted notebook **hart-do.nb**.

```
In[2]:= implies[member[x, V], member[hartogs[x], V]]

Out[2]= True
```

The strategy will be to replace the proper class **Ω** in the above intersection by the set **hartogs[x] = Ω ⋂ image[Q, P[x]]**.

derivation

The derivation of the following theorem is speeded up by omitting the three proof steps that have been commented out with **(* … *)**.

Theorem. If **w ∈ BIJ ⋂ image[inverse[IMAGE[FIRST]], Ω] ⋂ P[V × x]**, then **domain[w]** is equipollent to a subclass of **x**.

```
In[3]:= Map[not, SubstTest[and, (*implies[p1,p4],*) implies[and[p1, p4], p5],
      (*implies[p1,p6],implies[p1,p7], *) implies[and[p5, p6, p7], p8],
      not[implies[p1, p8]], {p1 → and[member[w, map[domain[w], x]],
         FUNCTION[inverse[w]], member[domain[w], OMEGA]], p4 → FUNCTION[w],
       p5 -> member[pair[range[w], domain[w]], Q], p6 → member[domain[w], OMEGA],
       p7 → subclass[range[w], x], p8 → member[domain[w], image[Q, P[x]]]}]] // Reverse

Out[3]= or[member[domain[w], image[Q, P[x]]], not[FUNCTION[inverse[w]]],
     not[member[w, map[domain[w], x]]], not[member[domain[w], OMEGA]]] == True

In[4]:= (% /. {w → w_, x → x_}) /. Equal → SetDelayed
```

A series of lemmas is used to derive the main result.

Lemma. The class **Ω ⋂ image[IMAGE[FIRST], BIJ⋂ P[V× x]]** is a subclass of **hartogs[x]**.

```
In[5]:= Map[equal[V, #] &, SubstTest[class, t, implies[member[t, u], member[t, v]],
     {u → intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]],
      v → intersection[BIJ, image[inverse[IMAGE[FIRST]],
         intersection[OMEGA, image[Q, P[x]]]], P[cart[V, x]]]}]]

Out[5]= subclass[intersection[OMEGA, image[IMAGE[FIRST], intersection[BIJ, P[cart[V, x]]]]],
     image[Q, P[x]]] == True

In[6]:= subclass[intersection[OMEGA, image[IMAGE[FIRST], intersection[BIJ, P[cart[V, x_]]]]],
     image[Q, P[x_]]] := True
```

Theorem. An equation alluded to in the introductory remarks.

```
In[7]:= SubstTest[and, subclass[u, v], subclass[v, u],
    {u → intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]],
     v → intersection[BIJ,
       image[inverse[IMAGE[FIRST]], intersection[OMEGA, image[Q, P[x]]]], P[cart[V, x]]]}]

Out[7]= equal[intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]],
     intersection[BIJ, image[inverse[IMAGE[FIRST]], intersection[OMEGA, image[Q, P[x]]]],
      P[cart[V, x]]]] == True

In[8]:= intersection[BIJ,
     image[inverse[IMAGE[FIRST]], intersection[OMEGA, image[Q, P[x_]]]], P[cart[V, x_]]] :=
    intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]]
```

Lemma. A technical lemma based on the idea that any class **u** is contained in the power class **P[U[u]]**.

```
In[23]:= SubstTest[implies, subclass[x, z], subclass[image[t, x], image[t, z]],
      {t -> composite[id[intersection[BIJ, P[cart[V, y]]]], inverse[IMAGE[FIRST]]],
       z → P[U[x]]}] // Reverse

Out[23]= subclass[core[intersection[BIJ, image[inverse[IMAGE[FIRST]], x]], cart[V, y]],
      cart[U[x], y]] == True

In[24]:= (% /. {x → x_, y → y_}) /. Equal → SetDelayed
```

Main Theorem. The class **BIJ ⋂ image[inverse[IMAGE[FIRST]], Ω] ⋂ P[V × setpart[x]]** is a set.

```
In[28]:= (SubstTest[implies, and[subclass[u, v], member[v, V]], member[u, V],
           {u -> intersection[BIJ, image[inverse[IMAGE[FIRST]], t], P[cart[V, setpart[x]]]],
            v → P[cart[U[t], setpart[x]]]}] /. t -> hartogs[setpart[x]]) // Reverse

Out[28]= member[intersection[BIJ,
            image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, setpart[x]]]], V] == True

In[29]:= member[intersection[BIJ,
            image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, setpart[x_]]]], V] := True
```

Corollary. (Remove the **setpart** wrapper.)

```
In[31]:= Map[implies[member[x, y], #] &,
          SubstTest[implies, equal[x, setpart[t]], member[intersection[BIJ,
             image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]], V], t → x]] // Reverse

Out[31]= or[member[intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x]]], V],
          not[member[x, y]]] == True

In[32]:= or[member[intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA], P[cart[V, x_]]], V],
          not[member[x_, y_]]] := True
```

The following is a variable-free restatement of the main theorem.

Corollary. The inverse of the restriction of **IMAGE[SECOND]** to the class of bijections whose domains are ordinals is thin.

```
In[38]:= Map[equal[V, domain[#]] &,
          SubstTest[reify, x, image[V, set[intersection[t, image[inverse[IMAGE[FIRST]], OMEGA],
             P[cart[V, setpart[x]]]]]], t → BIJ]]

Out[38]= equal[V, domain[
            VERTSECT[composite[id[intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA]]],
             inverse[IMAGE[SECOND]]]]]] == True

In[40]:= domain[VERTSECT[composite[id[intersection[BIJ, image[inverse[IMAGE[FIRST]], OMEGA]]],
           inverse[IMAGE[SECOND]]]]] := V
```