Hartogs number for finite sets

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```
In[1]:= SetDirectory["l:"]; << goedel.08jan18a; << tools.m

        :Package Title: goedel.08jan18a                2008 January 18 at 7:45 p.m.

        It is now:  2008 Jan 20 at 22:55

        Loading Simplification Rules

        TOOLS.M                                        Revised 2008 January 2

        weightlimit = 40
```

introduction

The **Hartogs number** of a class **x** is defined to be the class of all ordinals which are equipollent to some subset of **x**.

```
In[2]:= class[y, and[member[y, OMEGA], exists[z, and[member[pair[y, z], Q], subclass[z, x]]]]]

Out[2]= intersection[OMEGA, image[Q, P[x]]]
```

In this notebook an explicit formula for the Hartogs number is derived for the special case that **x** is a finite set.

derivation

Lemma.

```
In[3]:= IminComp[CARD, id[omega], P[nat[x]]] // Reverse

Out[3]= intersection[omega, image[inverse[CARD], P[nat[x]]]] == succ[nat[x]]

In[4]:= intersection[omega, image[inverse[CARD], P[nat[x_]]]] := succ[nat[x]]
```

Lemma.

```
In[5]:= Map[intersection[OMEGA, #] &, IminComp[CARD, id[FINITE], P[nat[x]]]]

Out[5]= intersection[OMEGA, image[inverse[CARD], succ[nat[x]]]] == succ[nat[x]]

In[6]:= intersection[OMEGA, image[inverse[CARD], succ[nat[x_]]]] := succ[nat[x]]
```

Lemma.

```
In[7]:=  equal[intersection[FINITE, P[nat[x]]], P[nat[x]]]

Out[7]=  True

In[8]:=  intersection[FINITE, P[nat[x_]]] := P[nat[x]]
```

Lemma. (Hartogs number for a natural number.)

```
In[9]:=  Map[intersection[OMEGA, #] &, ImageComp[Q, id[FINITE], P[nat[x]]]] // Reverse

Out[9]=  intersection[OMEGA, image[Q, P[nat[x]]]] == succ[nat[x]]

In[10]:=  intersection[OMEGA, image[Q, P[nat[x_]]]] := succ[nat[x]]
```

Lemma.

```
In[11]:=  SubstTest[intersection, OMEGA,
            image[inverse[CARD], succ[nat[t]]], t → card[fin[x]]] // Reverse

Out[11]=  intersection[OMEGA, image[inverse[CARD], succ[card[fin[x]]]]] == succ[card[fin[x]]]

In[12]:=  intersection[OMEGA, image[inverse[CARD], succ[card[fin[x_]]]]] := succ[card[fin[x]]]
```

Theorem. (Hartogs number for a finite set.)

```
In[13]:=  Map[intersection[OMEGA, #] &, ImageComp[Q, id[FINITE], P[fin[x]]]] // Reverse

Out[13]=  intersection[OMEGA, image[Q, P[fin[x]]]] == succ[card[fin[x]]]

In[14]:=  intersection[OMEGA, image[Q, P[fin[x_]]]] := succ[card[fin[x]]]
```