Hartogs number

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```
In[1]:= SetDirectory["l:"]; << goedel.08jan18a; << tools.m

        :Package Title: goedel.08jan18a                    2008 January 18 at 7:45 p.m.

        It is now:  2008 Jan 20 at 23:48

        Loading Simplification Rules

        TOOLS.M                                            Revised 2008 January 2

        weightlimit = 40
```

introduction

The **Hartogs number** of a class **x** is defined to be the class of all ordinals which are equipollent to some subset of **x**. In this notebook it is shown that the Hartogs number is either an ordinal number or the class **OMEGA** of all ordinals. Equipollent sets have the same Hartogs number.

derivation

Lemma.

```
In[2]:= SubstTest[implies, subclass[u, v], subclass[image[w, u], image[w, v]],
            {u -> intersection[OMEGA, image[Q, P[x]]],
                v -> image[Q, P[x]], w -> inverse[S]}] // Reverse

Out[2]= subclass[image[inverse[S], intersection[OMEGA, image[Q, P[x]]]], image[Q, P[x]]] == True

In[3]:= subclass[
          image[inverse[S], intersection[OMEGA, image[Q, P[x_]]]], image[Q, P[x_]]] := True
```

Theorem. The Hartogs number of a class is an ordinal or the class of all ordinals.

```
In[4]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
            {u -> U[intersection[OMEGA, image[Q, P[x]]]],
                v -> image[inverse[S], intersection[OMEGA, image[Q, P[x]]]],
                w -> image[Q, P[x]]}] // Reverse

Out[4]= or[member[intersection[OMEGA, image[Q, P[x]]], OMEGA],
          subclass[OMEGA, image[Q, P[x]]]] == True
```

```
In[5]:= or[member[intersection[OMEGA, image[Q, P[x_]]], OMEGA],
          subclass[OMEGA, image[Q, P[x_]]]] := True
```

equipollence results

Theorem. Equipollent sets have equipollent power sets.

```
In[6]:= SubstTest[implies, and[member[u, v], subclass[v, w]], member[u, w],
           {u → pair[x, y], v → Q, w → composite[inverse[POWER], Q, POWER]}] // Reverse // MapNotNot

Out[6]= or[member[pair[P[x], P[y]], Q], not[member[pair[x, y], Q]]] == True

In[7]:= or[member[pair[P[x_], P[y_]], Q], not[member[pair[x_, y_], Q]]] := True
```

Lemma.

```
In[8]:= Map[implies[and[member[x, V], member[y, V]], #] &,
         SubstTest[implies, equal[u, v], equal[image[t, u], image[t, v]],
           {t → inverse[S], u -> image[Q, set[x]], v -> image[Q, set[y]]}] // Reverse]

Out[8]= or[equal[image[Q, P[x]], image[Q, P[y]]], not[equal[image[Q, set[x]], image[Q, set[y]]]],
          not[member[x, V]], not[member[y, V]]] == True

In[9]:= (% /. {x → x_, y → y_}) /. Equal → SetDelayed
```

Theorem.

```
In[10]:= Map[not, SubstTest[and, implies[p1, p2],
             implies[p1, p3], implies[p1, p4], implies[and[p2, p3, p4], p5],
             not[implies[p1, p5]], {p1 -> member[pair[x, y], Q], p2 -> member[x, V],
              p3 -> member[y, V], p4 -> equal[image[Q, set[x]], image[Q, set[y]]],
              p5 -> equal[image[Q, P[x]], image[Q, P[y]]]}]] // Reverse

Out[10]= or[equal[image[Q, P[x]], image[Q, P[y]]], not[member[pair[x, y], Q]]] == True

In[11]:= or[equal[image[Q, P[x_]], image[Q, P[y_]]], not[member[pair[x_, y_], Q]]] := True
```

Corollary. Equipollent sets have the same Hartogs number.

```
In[12]:= Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3], not[implies[p1, p3]],
             {p1 -> member[pair[x, y], Q], p2 -> equal[image[Q, P[x]], image[Q, P[y]]],
              p3 -> equal[intersection[OMEGA, image[Q, P[x]]],
                intersection[OMEGA, image[Q, P[y]]]]}]] // Reverse

Out[12]= or[equal[intersection[OMEGA, image[Q, P[x]]], intersection[OMEGA, image[Q, P[y]]]],
           not[member[pair[x, y], Q]]] == True

In[13]:= or[equal[intersection[OMEGA, image[Q, P[x_]]], intersection[OMEGA, image[Q, P[y_]]]],
           not[member[pair[x_, y_], Q]]] := True
```