images of subgroup ranges under homomorphisms

Johan G. F. Belinfante
2012 February 14

```
In[1]:= SetDirectory["l:"]; << goedel.12feb14a

        :Package Title: goedel.12feb14a              2012 February 14 at 1:50 p.m.

        Loading takes about thirteen minutes, half that time due to builtin pauses.

        It is now:  2012 Feb 14 at 15:35

        Loading Simplification Rules

        TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

        weightlimit = 40

        Loading completed.

        It is now:  2012 Feb 14 at 15:49
```

summary

The image of the range of a subgroup of a group **x** under a binary homomorphism from **x** to a group **y** is the range of a subgroup of **y**.

derivation

Lemma.

```
In[5]:= SubstTest[implies, and[member[v, GROUPS], member[y, GROUPS], member[u, binhom[v, y]]],
          member[range[u], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]],
          {u → composite[t, id[w]], v → composite[x, id[cart[w, w]]]}] // Reverse

Out[5]= or[member[image[t, w], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]],
          not[member[y, GROUPS]],
          not[member[composite[t, id[w]], binhom[composite[x, id[cart[w, w]]], y]]],
          not[member[composite[x, id[cart[w, w]]], GROUPS]]] == True

In[6]:= (% /. {t → t_, w → w_, x → x_, y → y_}) /. Equal → SetDelayed
```

Theorem. If **t** is a binary homomorphism from a group **x** to a group **y** and if **w** is the range of a subgroup of **x**, then **image[t, w]** is the range of a subgroup of **y**.

```
In[14]:= Map[not, SubstTest[and, (*implies[and[p1,p2],p3],implies[and[p1,p2],p4],*) implies[
           and[p1, p3, p4], p5], implies[and[p1, p3, p5], p6], not[implies[and[p1, p2], p6]],
          {p1 → and[member[x, GROUPS], member[y, GROUPS], member[t, binhom[x, y]]],
           p2 -> member[w, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
           p3 -> member[composite[x, id[cart[w, w]]], GROUPS],
           p4 -> member[id[w], binhom[composite[x, id[cart[w, w]]], x]], p5 ->
            member[composite[t, id[w]], binhom[composite[x, id[cart[w, w]]], y]], p6 → member[
             image[t, w], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]}]] // Reverse

Out[14]= or[member[image[t, w], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]],
          not[member[t, binhom[x, y]]],
          not[member[w, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
          not[member[x, GROUPS]], not[member[y, GROUPS]]] == True

In[16]:= or[member[image[t_, w_], image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]],
          not[member[t_, binhom[x_, y_]]],
          not[member[w_, image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]]],
          not[member[x_, GROUPS]], not[member[y_, GROUPS]]] := True
```

Lemma. The variable **w** can be eliminated. The **funpart** wrapper is used to convert an inverse image to a direct image.

```
In[19]:= (Map[equal[V, domain[#]] &, SubstTest[reify, w,
           case[or[member[image[u, w], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]], not[
              member[u, z]], not[member[w, image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
             not[member[x, GROUPS]], not[member[y, GROUPS]]]],
           z → binhom[x, y]]]) /. u → funpart[t]

Out[19]= or[not[member[x, GROUPS]],
          not[member[y, GROUPS]], not[member[funpart[t], binhom[x, y]]],
          subclass[image[IMAGE[funpart[t]], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
           image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]] == True

In[20]:= (% /. {t → t_, x → x_, y → y_}) /. Equal → SetDelayed
```

Lemma. Eliminate the **funpart** wrapper. (This introduces a redundant literal.)

```
In[22]:= SubstTest[implies, equal[t, funpart[w]],
          or[not[member[x, GROUPS]], not[member[y, GROUPS]], not[member[t, binhom[x, y]]],
           subclass[image[IMAGE[t], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
            image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]], w → t] // Reverse

Out[22]= or[not[FUNCTION[t]], not[member[t, binhom[x, y]]],
          not[member[x, GROUPS]], not[member[y, GROUPS]],
          subclass[image[IMAGE[t], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
           image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]] == True

In[23]:= (% /. {t → t_, x → x_, y → y_}) /. Equal → SetDelayed
```

Finally, the redundant literal **FUNCTION[t]** is eliminated in the usual fashion.

Theorem. If **t** is a binary homomorphism from a group **x** to a group **y**, then **IMAGE[t]** conducts the class of ranges of subgroups of **x** to the class of ranges of subgroups of **y**.

```
In[25]:= SubstTest[and, implies[p, q], or[p, q], {p → FUNCTION[t],
           q -> or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]], not[member[y, GROUPS]],
             subclass[image[IMAGE[t], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
              image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]]}]

Out[25]= or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]], not[member[y, GROUPS]],
           subclass[image[IMAGE[t], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
            image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]] == True

In[27]:= or[not[member[t_, binhom[x_, y_]]], not[member[x_, GROUPS]], not[member[y_, GROUPS]],
           subclass[image[IMAGE[t_], image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]],
            image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]]] := True
```

The variable **t** can also be eliminated.

Theorem. Images of ranges of subgroups of a group **x** under binary homomorphisms from **x** to a group **y** are ranges of subgroups of **y**.

```
In[30]:= Map[equal[V, domain[#]] &,
          SubstTest[reify, t, case[or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]],
             not[member[y, GROUPS]], subclass[image[IMAGE[t], u], v]]],
           {u -> image[IMAGE[SECOND], intersection[GROUPS, P[x]]],
            v -> image[IMAGE[SECOND], intersection[GROUPS, P[y]]]}]]

Out[30]= or[not[member[x, GROUPS]], not[member[y, GROUPS]], subclass[
            image[IMG, cart[binhom[x, y], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
            image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]] == True

In[32]:= or[not[member[x_, GROUPS]], not[member[y_, GROUPS]], subclass[
            image[IMG, cart[binhom[x_, y_], image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]]],
            image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]]] := True
```