IMAGE[VERTSECT[DIV]]

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In[1]:=
```
SetDirectory["l:"]; << goedel94.12a; << tools.m
```

```
:Package Title: goedel94.12a      2007 June 12 at 4:20 p.m.

It is now:  2007 Jun 13 at 0:46

Loading Simplification Rules

TOOLS.M                                   Revised 2007 June 10

weightlimit = 40
```

summary

Every thin partial order is isomorphic to the restriction of the partial order **inverse[S]** to its vertical sections. For the special case of the divisibility relation on the natural numbers, this isomorphism can be described as a restriction of the total function **VERTSECT[DIV]**. This function assigns to each natural number **x** the set **image[DIV, set[x]]** of all multiples of **x**. If **x** is not a number, the function **VERTSECT[DIV]** assigns the empty set **0**. In either case, the value assigned to **x** can be described as the vertical section of the divisibility relation **DIV** at **x**. The restriction of the function **VERTSECT[DIV]** to the set **omega** of natural numbers is an isomorphism from the partial ordering of natural numbers by divisibility to the partial ordering of the vertical sections of **DIV** by (reversed) inclusion. In this notebook, a different, but closely related total function **IMAGE[VERTSECT[DIV]]** is also encountered. This function assigns to any set the set of all vertical sections for its members. The restriction of this function to subsets of the set **omega** of natural numbers is one-to-one, and satisfies other interesting properties.

one-to-one property

Theorem.

In[2]:=
```
SubstTest[FUNCTION, composite[id[P[domain[oopart[t]]]], inverse[IMAGE[oopart[t]]]],
   t → composite[VERTSECT[DIV], id[omega]]] // Reverse
```

Out[2]=
```
FUNCTION[composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]]]] == True
```

In[3]:=
```
FUNCTION[composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]]]] := True
```

Corollary.

```
In[4]:= SubstTest[composite, funpart[t], inverse[funpart[t]],
          t -> composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]]]] // Reverse

Out[4]= composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]],
          IMAGE[VERTSECT[DIV]], id[P[omega]]] == id[P[omega]]

In[5]:= composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]],
          IMAGE[VERTSECT[DIV]], id[P[omega]]] := id[P[omega]]
```

chains[DIV]

Theorem. Chains in the poset **omega** partially ordered by **DIV** correspond to chains in the poset **image[VERTSECT[DIV], omega]** of vertical sections partially ordered by **inverse[S]**.

```
In[6]:= SubstTest[chains, composite[inverse[funpart[t]], s, funpart[t]],
         {s → inverse[S], t → composite[VERTSECT[DIV], id[omega]]}]

Out[6]= intersection[image[inverse[IMAGE[VERTSECT[DIV]]], chains[S]], P[omega]] == chains[DIV]

In[7]:= intersection[image[inverse[IMAGE[VERTSECT[DIV]]], chains[S]], P[omega]] := chains[DIV]
```

Lemma.

```
In[9]:= equal[intersection[chains[DIV], P[omega]], chains[DIV]]

Out[9]= True

In[11]:= intersection[chains[DIV], P[omega]] := chains[DIV]
```

Corollary.

```
In[13]:= Map[subclass[#, chains[S]] &, ImageComp[composite[IMAGE[VERTSECT[DIV]], id[P[omega]]],
            composite[id[P[omega]], inverse[IMAGE[VERTSECT[DIV]]]], chains[S]]] // Reverse

Out[13]= subclass[image[IMAGE[VERTSECT[DIV]], chains[DIV]], chains[S]] == True

In[14]:= subclass[image[IMAGE[VERTSECT[DIV]], chains[DIV]], chains[S]] := True
```

domain[GREATEST[DIV]]

Theorem. If a set of natural numbers has a greatest element with respect to the divisibility relation, then that greatest element corresponds to the least element of the corresponding set of vertical sections. Accordingly, there is a relation between the sets of numbers which have greatest elements and the sets of vertical sections which have least elements.

```
In[15]:= SubstTest[domain, LEAST[composite[inverse[funpart[t]], S, funpart[t]]],
          t → composite[VERTSECT[DIV], id[omega]]]

Out[15]= intersection[image[inverse[IMAGE[VERTSECT[DIV]]], fix[composite[E, BIGCAP]]],
           P[omega]] == domain[GREATEST[DIV]]
```

```
In[16]:= intersection[image[inverse[IMAGE[VERTSECT[DIV]]], fix[composite[E, BIGCAP]]],
           P[omega]] := domain[GREATEST[DIV]]
```

Lemma.

```
In[18]:= equal[intersection[domain[GREATEST[DIV]], P[omega]], domain[GREATEST[DIV]]]

Out[18]= True

In[20]:= intersection[domain[GREATEST[DIV]], P[omega]] := domain[GREATEST[DIV]]
```

Theorem.

```
In[22]:= Map[subclass[#, fix[composite[E, BIGCAP]]] &,
           ImageComp[composite[IMAGE[VERTSECT[DIV]], id[P[omega]]], composite[id[P[omega]],
             inverse[IMAGE[VERTSECT[DIV]]]]], fix[composite[E, BIGCAP]]]] // Reverse

Out[22]= subclass[image[IMAGE[VERTSECT[DIV]], domain[GREATEST[DIV]]],
           fix[composite[E, BIGCAP]]] == True

In[23]:= subclass[image[IMAGE[VERTSECT[DIV]], domain[GREATEST[DIV]]],
           fix[composite[E, BIGCAP]]] := True
```