a theorem about constructible classes

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```
In[1]:= SetDirectory["l:"]; << goedel.13nov19a
```

```
:Package Title: goedel.13nov19a                 2013 November 19 at 4:10 p.m.

Loading takes about seventeen minutes, half that time due to builtin pauses.

It is now:  2013 Nov 20 at 5:56

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2013 Nov 20 at 6:13
```

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summary

A rewrite rule is derived which generalizes a theorem about constructible classes stated on page 148 in the following reference.

"Raymond M. Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem*, Oxford Science Publications, Clarendon Press, Oxford, 1996.";

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derivation

Theorem.

```
In[2]:= Map[not, SubstTest[and, implies[and[p1, p2], p4],
          implies[and[p3, p4], p5], not[implies[and[p1, p2, p3], p5]], {p1 -> member[x, w],
           p2 -> subclass[w, image[inverse[S], v]], p3 -> subclass[image[IMAGE[id[x]], v], z],
           p4 -> member[x, image[inverse[S], v]], p5 → member[x, z]}]] // Reverse

Out[2]= or[member[x, z], not[member[x, w]], not[subclass[w, image[inverse[S], v]]],
         not[subclass[image[IMAGE[id[x]], v], z]]] == True
```

```
In[3]:= or[member[x_, z_], not[member[x_, w_]], not[subclass[w_, image[inverse[S], v_]]],
         not[subclass[image[IMAGE[id[x_]], v_], z_]]] := True
```

Corollary. (Replace a variable with a power class.)

```
In[4]:= Map[implies[member[x, t], #] &,
          SubstTest[or, member[x, z], not[member[x, w]], not[subclass[w, image[inverse[S], v]]],
           not[subclass[image[IMAGE[id[x]], v], z]], w → P[y]]] // Reverse

Out[4]= or[member[x, z], not[member[x, t]],
          not[subclass[x, y]], not[subclass[image[IMAGE[id[x]], v], z]],
          not[subclass[P[y], image[inverse[S], v]]]] == True

In[5]:= or[member[x_, z_], not[member[x_, t_]],
          not[subclass[x_, y_]], not[subclass[image[IMAGE[id[x_]], v_], z_]],
          not[subclass[P[y_], image[inverse[S], v_]]]] := True
```

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remarks

Kurt Gödel introduced a class **L** of constructible sets as a model of his class theory. A set **x** is a **constructible set** if **x ∈ L**. A class **x** is a **constructible class** if **x ⊂ L** and **image[IMAGE[id[x]], L] ⊂ L**. The latter condition says that the intersection of **x** with any constructible set is a constructible set.

```
In[9]:= assert[forall[t, implies[member[t, L], member[intersection[t, x], L]]]]

Out[9]= subclass[image[IMAGE[id[x]], L], L]
```

It can be shown that **L** is **almost universal** in the sense that **P[L] ⊂ image[inverse[S], L]**. That is, every subset of **L** is a subset of a constructible set.

```
In[8]:= assert[forall[s, implies[subclass[s, L], exists[t, and[subclass[s, t], member[t, L]]]]]]

Out[8]= subclass[P[L], image[inverse[S], L]]
```

The corollary derived in the preceding section implies the following, which is stated as Proposition 3.2 in the book of Smullyan and Fitting.

```
In[6]:= implies[and[subclass[P[L], image[inverse[S], L]],
          member[x, V], subclass[x, L], invariant[IMAGE[id[x]], L]], member[x, L]]

Out[6]= True
```

This result says that if a set is a constructible class, then it is a constructible set.