the category of sets is locally small

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```
In[1]:= SetDirectory["l:"]; << goedel.09aug16a;<< tools.m

        :Package Title: goedel.09aug16a            2009 August 16 at 11:55 a.m.

        It is now:  2009 Aug 17 at 17:25

        Loading Simplification Rules

        TOOLS.M                                    Revised 2009 July 2

        weightlimit = 40
```

summary

An (arrows-only) category **cat[x]** is said to be **locally small** if for every pair of identity morphisms **u, v ∈ ids[cat[x]]**, the class of morphisms from **u** to **v** is a set. It is convenient to reformulate this condition in a fashion that avoids the explicit quantification over the set variables **u** and **v**. The key to hiding these quantifiers is to make use of the ternary relation **hom[cat[x]]** which by definition consists of all ordered triples **pair[pair[u, v], w]** such that **w** is a morphism from **u** to **v**. In other words, the membership statement **pair[pair[u, v], w] ∈ hom[cat[x]]** holds if and only if the morphism **w ∈ range[cat[x]]** satisfies the equations **u = APPLY[dom[cat[x]], w]** and **v = APPLY[cod[cat[x]], w]**. The assertion that a category **cat[x]** is locally small therefore translates into the condition that every vertical section **image[hom[cat[x]], set[PAIR[u,v]]]** of the ternary relation **hom[cat[x]]** is a set. Equivalently, a category **cat[x]** is locally small if and only if the corresponding ternary relation **hom[cat[x]]** is thin, which can be stated concisely as the equation **domain[VERTSECT[hom[cat[x]]] = V**. In this notebook it is shown that the category of sets **CATOFUNS** is locally small.

derivation

In the category of sets each identity morphism is of the form **PAIR[id[u], u]**, where **u** is a set. Each vertical section of the ternary relation **hom[CATOFUNS]** can be expressed as follows in terms of the set **map[u, v]** of mappings from **u** to **v**.

Theorem. An explicit formula for a typical vertical section of the ternary relation **hom[CATOFUNS]**.

```
In[2]:= IminComp[cross[dom[t], cod[t]], DUP,
          set[PAIR[PAIR[id[u], u], PAIR[id[v], v]]]] /. t → CATOFUNS

Out[2]= image[hom[CATOFUNS], cart[cart[set[id[u]], set[u]], cart[set[id[v]], set[v]]]] ==
         cart[map[u, v], set[v]]

In[3]:= image[hom[CATOFUNS], cart[cart[set[id[u_]], set[u_]], cart[set[id[v_]], set[v_]]]] :=
         cart[map[u, v], set[v]]
```

This can be cleaned up by making the use of the function which takes each set to the corresponding identity morphism:

```
In[4]:= VERTSECT[reify[u, PAIR[id[u], u]]]

Out[4]= composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]]
```

Lemma. (Eliminating the variables **u** and **v** from the statement that the vertical section at any pair of identity morphisms is a set.)

```
In[5]:= Map[empty[composite[Id, complement[#]]] &,
         SubstTest[class, pair[u, v], member[image[t, set[PAIR[u, v]]], V], t ->
           composite[hom[CATOFUNS], cross[composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]],
             composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]]]]]]

Out[5]= subclass[cart[inverse[IMAGE[DUP]], inverse[IMAGE[DUP]]],
          domain[VERTSECT[hom[CATOFUNS]]]] == True

In[6]:= % /. Equal → SetDelayed
```

Theorem. The category of sets is locally small.

```
In[7]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
          subclass[u, w], {u → domain[hom[CATOFUNS]], v → cartsq[ids[CATOFUNS]],
           w → domain[VERTSECT[hom[CATOFUNS]]]}] // Reverse

Out[7]= equal[V, domain[VERTSECT[hom[CATOFUNS]]]] == True

In[8]:= domain[VERTSECT[hom[CATOFUNS]]] := V
```