monotone relations are functions

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```
In[1]:= SetDirectory["l:"]; << goedel.10aug06a; << tools.m

        :Package Title: goedel.10aug06a            2010 August 6 at 8:00 p.m.

        It is now:  2010 Aug 7 at 8:38

        Loading Simplification Rules

        TOOLS.M                                    Revised 2010 July 26

        weightlimit = 40
```

summary

Monotone relations are functions. This interesting fact was discovered in the course of trying to understand a recently derived theorem about monotone functions. A generalization of the following recently derived theorem is obtained, dropping the hypothesis that **x** be a function.

```
In[85]:= implies[and[FUNCTION[x], subclass[composite[x, S, inverse[x]], S]],
          subclass[x, composite[S, CORE[fix[x]]]]]

Out[85]= True
```

It was then discovered that the appearance of having generalized the existing theorem is in fact illusory because the monotonicity hypothesis implies that **Id** ∘ **x** is a function.

derivation

Lemma.

```
In[76]:= SubstTest[implies, subclass[u, v], subclass[composite[u, w], composite[v, w]],
           {u -> composite[x, S, id[y]], v -> S, w → E}] // Reverse

Out[76]= or[not[subclass[composite[x, S, id[y]], S]],
           subclass[composite[x, inverse[CORE[y]]], S]] == True

In[77]:= or[not[subclass[composite[x_, S, id[y_]], S]],
           subclass[composite[x_, inverse[CORE[y_]]], S]] := True
```

Lemma.

```
In[79]:= or[not[subclass[composite[x, S, id[y]], S]],
           subclass[composite[Id, x], composite[S, CORE[y]]]] // AssertTest

Out[79]= or[not[subclass[composite[x, S, id[y]], S]],
           subclass[composite[Id, x], composite[S, CORE[y]]]] == True

In[80]:= or[not[subclass[composite[x_, S, id[y_]], S]],
           subclass[composite[Id, x_], composite[S, CORE[y_]]]] := True
```

Theorem.

```
In[83]:= Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3],
            not[implies[p1, p3]], {p1 → subclass[composite[x, S, inverse[x]], S],
             p2 -> subclass[composite[x, S, id[fix[x]]], S],
             p3 -> subclass[composite[Id, x], composite[S, CORE[fix[x]]]]}]] // Reverse

Out[83]= or[not[subclass[composite[x, S, inverse[x]], S]],
           subclass[composite[Id, x], composite[S, CORE[fix[x]]]]] == True

In[84]:= or[not[subclass[composite[x_, S, inverse[x_]], S]],
           subclass[composite[Id, x_], composite[S, CORE[fix[x_]]]]] := True
```

The appearance of having generalized the existing theorem is illusory because the monotonicity requirement implies that **Id ∘ x** is a function. This fact will now be derived.

Theorem.

```
In[92]:= SubstTest[subclass, t, intersection[u, v],
          {t → composite[x, inverse[x]], u → S, v → inverse[S]}]

Out[92]= subclass[composite[x, inverse[x]], S] == FUNCTION[composite[Id, x]]

In[93]:= subclass[composite[x_, inverse[x_]], S] := FUNCTION[composite[Id, x]]
```

Theorem.

```
In[96]:= SubstTest[implies, subclass[composite[x, S, inverse[x]], t],
           subclass[composite[x, inverse[x]], t], t → S] // Reverse

Out[96]= or[FUNCTION[composite[Id, x]], not[subclass[composite[x, S, inverse[x]], S]]] == True

In[97]:= or[FUNCTION[composite[Id, x_]], not[subclass[composite[x_, S, inverse[x_]], S]]] := True
```

Theorem. A variable-free restatement of the theorem that monotone relations are functions.

```
In[104]:=
       Map[equal[V, #] &, dif[cliques[complement[cross[S, complement[S]]]],
           image[inverse[IMAGE[id[cart[V, V]]]], FUNS]] // complement // Normality]

Out[104]=
       subclass[
         intersection[cliques[complement[cross[S, complement[S]]]], P[cart[V, V]]], FUNS] == True
```

In[105]:=

```
subclass[intersection[
   cliques[complement[cross[S, complement[S]]]], P[cart[V, V]]], FUNS] := True
```