P[FINITE]

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```
In[1]:= << goedel53.22c; << tools.m

        :Package Title: goedel53.22c      2004 January 22 at 8:00 p.m.

        It is now:  2004 Jan 26 at 12:53

        Loading Simplification Rules

        TOOLS.M                      Revised 2004 January 3

        weightlimit = 40
```

summary

The statement **member[x, FINITE]** says that **x** is a finite set, whereas **subclass[x, FINITE]** says that **x** is a collection of finite sets. These conditions are independent: a collection of finite sets need not be finite, and conversely. The class **FINITE** of finite sets and the class **P[FINITE]** of collections of finite sets are thus two different classes, neither of which is contained in the other. In this notebook some properties of the class **P[FINITE]** are derived. Of particular interest will be how this class relates to the operations **Aclosure** and **Uclosure**. For the class **FINITE**, the results are already known:

```
In[2]:= {member[Aclosure[x], FINITE], member[Uclosure[x], FINITE]}

Out[2]= {member[x, FINITE], member[x, FINITE]}
```

the hereditary closure image[inverse[S],x]

The hereditary closure **image[inverse[S], x]** is studied first because it will be needed to deduce the results for **Aclosure**. The only fact that will be needed for this is easy to derive:

```
In[3]:= subclass[image[inverse[S], x], FINITE] // AssertTest

Out[3]= subclass[image[inverse[S], x], FINITE] == subclass[x, FINITE]

In[4]:= subclass[image[inverse[S], x_], FINITE] := subclass[x, FINITE]
```

finiteness of the hereditary closure

In this section a condition is derived for the hereditary closure to be finite. These results are not needed for the rest of the notebook, but are nonetheless interesting in their own right. Some lemmas are required:

```
In[5]:= SubstTest[subclass, x, image[inverse[POWER], y], y -> FINITE] // Reverse

Out[5]= subclass[image[POWER, x], FINITE] == subclass[x, FINITE]

In[6]:= subclass[image[POWER, x_], FINITE] := subclass[x, FINITE]
```

The remaining lemmas are subsumed by the final result in this section.

```
In[7]:= Map[implies[#, subclass[x, FINITE]] &,
         SubstTest[member, U[y], FINITE, y -> image[POWER, x]]]

Out[7]= or[not[member[image[inverse[S], x], FINITE]], subclass[x, FINITE]] == True

In[8]:= (% /. x -> x_) /. Equal -> SetDelayed

In[9]:= SubstTest[implies, and[subclass[x, y], member[y, FINITE]],
         member[x, FINITE], y -> image[inverse[S], x]]

Out[9]= or[member[x, FINITE], not[member[image[inverse[S], x], FINITE]]] == True

In[10]:= (% /. x -> x_) /. Equal -> SetDelayed

In[11]:= SubstTest[implies, and[subclass[u, v], member[v, FINITE]],
          member[u, FINITE], {u -> image[inverse[S], x], v -> P[U[x]]}]

Out[11]= or[member[image[inverse[S], x], FINITE],
           not[member[x, FINITE]], not[subclass[x, FINITE]]] == True

In[12]:= (% /. x -> x_) /. Equal -> SetDelayed
```

An application of double negation is required to establish the following logical equivalence:

```
In[13]:= equiv[member[image[inverse[S], x], FINITE], member[U[x], FINITE]] // not // not

Out[13]= True
```

This justifies the following rewrite rule:

```
In[14]:= member[image[inverse[S], x_], FINITE] := and[member[x, FINITE], subclass[x, FINITE]]
```

Aclosure results

The condition for a hereditary closure to be a collection of finite sets is used to derive the condition for an Aclosure to be a collection of finite sets:

```
In[15]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
          subclass[u, w], {u -> Aclosure[x], v -> image[inverse[S], x], w -> FINITE}]

Out[15]= or[not[subclass[x, FINITE]], subclass[Aclosure[x], FINITE]] == True

In[16]:= (% /. x -> x_) /. Equal -> SetDelayed
```

The reverse implication also holds, so the following logical equivalence holds:

```
In[17]:= equiv[subclass[Aclosure[x], FINITE], subclass[x, FINITE]]

Out[17]= True
```

A rewrite rule can be introduced to express this fact.

```
In[18]:= subclass[Aclosure[x_], FINITE] := subclass[x, FINITE]
```

corollaries

Some variable−free corollaries can be deduced.

```
In[19]:= image[inverse[ACLOSURE], P[FINITE]] // Normality

Out[19]= image[inverse[ACLOSURE], P[FINITE]] == P[FINITE]

In[20]:= image[inverse[ACLOSURE], P[FINITE]] := P[FINITE]

In[21]:= ImageComp[ACLOSURE, inverse[ACLOSURE], P[FINITE]] // Reverse

Out[21]= image[ACLOSURE, P[FINITE]] == intersection[fix[ACLOSURE], P[FINITE]]

In[22]:= image[ACLOSURE, P[FINITE]] := intersection[fix[ACLOSURE], P[FINITE]]
```

comments about fix[HULL[x]]

The class **fix[HULL[x]]** is closely related to the class **Aclosure[x]**. They are equal when **x** is a set. For proper classes it is known only that **Aclosure[x]** is contained in **fix[HULL[x]]**. No proof is known for the reverse inclusion, nor have any counterexamples been found. Because of this unsettled state of affairs, a separate rule is needed; it is derived in the exactly the same way as the corresponding results for **Aclosure**:

```
In[23]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
          subclass[u, w], {u -> fix[HULL[x]], v -> image[inverse[S], x], w -> FINITE}]

Out[23]= or[not[subclass[x, FINITE]], subclass[fix[HULL[x]], FINITE]] == True

In[24]:= (% /. x -> x_) /. Equal -> SetDelayed

In[25]:= equiv[subclass[fix[HULL[x]], FINITE], subclass[x, FINITE]]

Out[25]= True

In[26]:= subclass[fix[HULL[x_]], FINITE] := subclass[x, FINITE]
```

Uclosures

The **Uclosure** of a class is not contained in the hereditary closure, so a different approach is needed. The following lemma suffices:

```
In[27]:= SubstTest[subclass, x, image[inverse[BIGCUP], y], y -> FINITE] // Reverse

Out[27]= subclass[image[BIGCUP, x], FINITE] == and[subclass[x, FINITE], subclass[U[x], FINITE]]

In[28]:= subclass[image[BIGCUP, x_], FINITE] := and[subclass[x, FINITE], subclass[U[x], FINITE]]
```

As a corollary, one finds:

```
In[29]:= SubstTest[subclass, image[BIGCUP, y], FINITE, y -> P[x]]

Out[29]= subclass[Uclosure[x], FINITE] == and[subclass[x, FINITE], subclass[P[x], FINITE]]

In[30]:= subclass[Uclosure[x_], FINITE] := and[subclass[x, FINITE], subclass[P[x], FINITE]]
```

Comment. When **x** is a set, the condition **subclass[P[x], FINITE]** is equivalent to **member[x,FINITE]**. The question of whether this also holds for proper classes is equivalent to the question whether a proper class can fail to have an infinite subset. This raises the related question of whether the **Uclosure** of a proper class could contain only finite sets. In other words, if the **Uclosure** of **x** is a collection of finite sets, does **x** have to be a set?

corollaries

A variable−free version of the above results are derived for the case of sets.

```
In[31]:= image[inverse[UCLOSURE], P[FINITE]] // Normality

Out[31]= image[inverse[UCLOSURE], P[FINITE]] == intersection[FINITE, P[FINITE]]

In[32]:= image[inverse[UCLOSURE], P[FINITE]] := intersection[FINITE, P[FINITE]]

In[33]:= ImageComp[UCLOSURE, inverse[UCLOSURE], P[FINITE]] // Reverse

Out[33]= image[UCLOSURE, intersection[FINITE, P[FINITE]]] ==
          intersection[fix[UCLOSURE], P[FINITE]]

In[34]:= image[UCLOSURE, intersection[FINITE, P[FINITE]]] :=
          intersection[fix[UCLOSURE], P[FINITE]]
```

The following corollary says that any Uclosed collection of finite sets is finite.

```
In[35]:= Map[subclass[image[UCLOSURE, #], FINITE] &, IminComp[BIGCUP, UCLOSURE, FINITE]]

Out[35]= subclass[intersection[fix[UCLOSURE], P[FINITE]], FINITE] == True

In[36]:= subclass[intersection[fix[UCLOSURE], P[FINITE]], FINITE] := True
```