x has no infinite subsets iff P[x] does not

Johan G. F. Belinfante
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```
In[1]:= SetDirectory["l:"]; << goedel79.01a; << tools.m

        :Package Title: goedel79.01a           2006 March 1 at 6:30 p.m.

        It is now:  2006 Mar 2 at 14:54

        Loading Simplification Rules

        TOOLS.M                      Revised 2006 February 3

        weightlimit = 40
```

summary

A class has no infinite subsets if and only if its power class has no infinite subsets.

derivation

If **y** is an infinite subset of **x**, then **P[y]** is an infinite subset of **P[x]**.

```
In[2]:= SubstTest[implies, subclass[u, v], subclass[image[w, u], image[w, v]],
          {u → P[P[x]], v → FINITE, w → inverse[POWER]}]

Out[2]= or[not[subclass[P[P[x]], FINITE]], subclass[P[x], FINITE]] == True

In[3]:= (% /. x → x_) /. Equal → SetDelayed
```

The converse also holds:

```
In[4]:= SubstTest[implies, subclass[u, v],
          subclass[image[w, u], image[w, v]], {u → P[x], v → FINITE, w → inverse[BIGCUP]}]

Out[4]= or[not[subclass[P[x], FINITE]], subclass[P[P[x]], FINITE]] == True

In[5]:= (% /. x → x_) /. Equal → SetDelayed
```

Combining these two results yields an interesting rewrite rule:

```
In[6]:= equiv[subclass[P[P[x]], FINITE], subclass[P[x], FINITE]]

Out[6]= True
```

```
In[7]:= subclass[P[P[x_]], FINITE] := subclass[P[x], FINITE]
```

the case of a set

When **x** is a set, the condition **subclass[P[x], FINITE]** is equivalent to the statement that **x** is finite.

```
In[8]:= SubstTest[implies, and[member[y, V], subclass[P[y], FINITE]],
          member[y, FINITE], y → setpart[x]]

Out[8]= or[member[setpart[x], FINITE], not[subclass[P[setpart[x]], FINITE]]] == True

In[9]:= (% /. x → x_) /. Equal → SetDelayed

In[10]:= equiv[subclass[P[setpart[x]], FINITE], member[setpart[x], FINITE]]

Out[10]= True

In[11]:= subclass[P[setpart[x_]], FINITE] := member[setpart[x], FINITE]
```

On account of this, the rewrite rule derived in the preceding section could be deduced for the case of sets from this fact:

```
In[12]:= member[P[x], FINITE]

Out[12]= member[x, FINITE]
```

a further comment

According to Harvey Friedman's response to a question posted by the author on the Foundations of Mathematics (FOM) newsgroup, the statement **subclass[P[x], FINITE]** is also equivalent to the statement that **x** is finite when one assumes the axiom of regularity. In the **GOEDEL** program, the axiom of regularity is not automatically assumed to be true, but one could consider adding the hypothesis that **x** be regular. This matter will not be pursued further here.