pair groupoid

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summary

The pair groupoid is shown to be an example of an associative relation. It is closely related to the rotation invariant function \( \text{RIF} \), which is not itself associative.

reference

http://mathworld.wolfram.com/Groupoid.html

definition of the pair groupoid

\begin{verbatim}
In[2]:= class[pair[pair[pair[u, v], pair[w, x]], pair[y, z]],
and[equal[y, u], equal[v, w], equal[x, z]]]


In[3]:= associative[%] // AssertTest


In[4]:= associative[composite[RIF, cross[SWAP, SWAP]]] := True

In[5]:= FUNCTION[composite[RIF, cross[SWAP, SWAP]]]

Out[5]= True
\end{verbatim}

The domain of this associative operation is not a cartesian product:

\begin{verbatim}
In[6]:= domain[composite[RIF, cross[SWAP, SWAP]]]

Out[6]= composite[inverse[FIRST], SECOND]
\end{verbatim}
the associated transitive relations

There are two transitive relations associated with any associative relation. In the present case there are:

\[\text{In[7]} := \text{implies[associative}[x], \text{TRANSITIVE}[\text{composite}[x, \text{inverse}[\text{FIRST}]])}\]
\[\text{Out[7]} := \text{True}\]
\[\text{In[8]} := \text{composite}[x, \text{inverse}[\text{FIRST}]] \text{/. } x \rightarrow \text{composite}[\text{RIF}, \text{cross}[\text{SWAP}, \text{SWAP}]]\]
\[\text{Out[8]} := \text{composite[inverse[FIRST], FIRST]}\]
\[\text{In[9]} := \text{implies[associative}[x], \text{TRANSITIVE}[\text{composite}[x, \text{inverse}[\text{SECOND}]])}\]
\[\text{Out[9]} := \text{True}\]
\[\text{In[10]} := \text{composite}[x, \text{inverse}[\text{SECOND}]] \text{/. } x \rightarrow \text{composite}[\text{RIF}, \text{cross}[\text{SWAP}, \text{SWAP}]]\]
\[\text{Out[10]} := \text{composite[inverse[SECOND], SECOND]}\]

The \textit{GOEDEL} program already knows these facts:

\[\text{In[11]} := \text{Map[TRANSITIVE,}\]
\[\quad \{\text{composite[inverse[FIRST], FIRST], composite[inverse[SECOND], SECOND]})\]\n\[\text{Out[11]} := \{\text{True, True}\}\]

\section*{RIF itself is not associative}

One might wonder whether the function \textit{RIF} itself is associative. It is not.

\[\text{In[12]} := \text{associative[\text{RIF}] // AssertTest}\]
\[\text{Out[12]} := \text{associative[\text{RIF}] == False}\]

The \textit{flip} of any associative relation is associative. For the pair groupoid, the \textit{flip} is:

\[\text{In[13]} := \text{associative[\text{composite[SWAP, RIF]]] // AssertTest}\]
\[\text{Out[13]} := \text{associative[\text{composite[SWAP, RIF]]} == \text{True}\]
\[\text{In[14]} := \text{associative[\text{composite[SWAP, RIF]]} := \text{True}\]

Some other examples about which one might be curious. The rotation of the pair groupoid is not associative:

\[\text{In[15]} := \text{associative[\text{rotate[composite[\text{RIF}, \text{cross[SWAP}, \text{SWAP}]]] // AssertTest}\]
\[\text{Out[15]} := \text{associative[\text{composite[SWAP, RIF, \text{cross[Id, SWAP]]}] == False}\]

Here is another rotation−invariant function, the \textit{flip} of \textit{RIF}. It is not associative, either.
\[ In[16]:= \text{rotate[composite[SWAP, RIF, cross[SWAP, SWAP]]]} \]
\[ Out[16]= \text{composite[SWAP, RIF, cross[SWAP, SWAP]]} \]

\[ In[17]:= \text{associative[flip[RIF]] // AssertTest} \]
\[ Out[17]= \text{associative[composite[SWAP, RIF, cross[SWAP, SWAP]]]} == \text{False} \]