Quaife's Theorems (Q13) and (Q14)

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```
In[1]:= SetDirectory["l:"]; << goedel91.24b; << tools.m

:Package Title: goedel91.24b        2007 March 24 at 12:25 noon

It is now:  2007 Mar 25 at 9:40

Loading Simplification Rules

TOOLS.M                        Revised 2007 March 3

weightlimit = 40
```

summary

In this notebook a rewrite rule is derived that implies Quaife's Theorem **(Q13)**. It was also discovered that Quaife's Theorem **(Q14)** only holds when the divisor **y** is nonzero.

```
In[2]:= "Art Quaife, Automated Development of Fundamental
           Mathematical Theories, Appendix 3. Theorems Proved in Peano's
           Arithmetic, Kluwer Academic Publishers, Dordrecht, 1992. Cf. p. 196";
```

derivation

Lemma. This is a consequence of the corollary to Quaife's Theorem **(Q1)**.

```
In[3]:= Map[not, SubstTest[implies, and[subclass[nat[t], nat[v]], subclass[nat[u], nat[w]]],
           subclass[natadd[nat[t], nat[u]], natadd[nat[v], nat[w]]],
           {t → natmod[nat[x], nat[z]], u → natmod[nat[y], nat[z]], v → nat[x], w → nat[y]}]] //
         Reverse

Out[3]= member[natadd[nat[x], nat[y]],
          natadd[natmod[nat[x], nat[z]], natmod[nat[y], nat[z]]]] == False

In[4]:= member[natadd[nat[x_], nat[y_]],
          natadd[natmod[nat[x_], nat[z_]], natmod[nat[y_], nat[z_]]]] := False
```

Corollary. This suffices to prove Quaife's Theorem **(Q13)**.

```
In[5]:= Map[not, SubstTest[implies,
          and[subclass[nat[u], nat[v]], member[nat[v], nat[w]]], member[nat[u], nat[w]],
          {u -> natadd[nat[x], nat[z]], v -> natadd[nat[x], nat[y], nat[z]],
           w -> natadd[natmod[nat[x], nat[y]], natmod[nat[z], nat[y]]]}]] // Reverse

Out[5]= member[natadd[nat[x], nat[y], nat[z]],
          natadd[natmod[nat[x], nat[y]], natmod[nat[z], nat[y]]]] == False

In[6]:= member[natadd[nat[x_], nat[y_], nat[z_]],
          natadd[natmod[nat[x_], nat[y_]], natmod[nat[z_], nat[y_]]]] := False
```

Quaife's Theorems (Q13) and (Q14)

Quaife's notation **x/y** is here interpreted as floored division:

```
In[7]:= natquot[x_, y_] := natdiv[natsub[x, natmod[x, y]], y]
```

The truth of Quaife's Theorem **(Q13)** is now recognized by the **GOEDEL** program.

```
In[8]:= example[q13, "implies[member[natadd[nat[x],nat[z]],
         natadd[nat[y],natmul[nat[y],natquot[nat[x],nat[y]]]],
         natmul[nat[y],natquot[nat[z],nat[
           y]]]]], equal[natquot[natadd[nat[x],nat[z]],nat[y]],
         natadd[natquot[nat[x],nat[y]], natquot[nat[z],nat[y]]]]]", ""]

       In[q13]:= implies[member[natadd[nat[x],nat[z]],  natadd[nat[y],natmul[nat[y],natquot[
          nat[x],nat[y]]]],  natmul[nat[y],natquot[nat[z],nat[y]]]]], equal[natquot[natadd[
          nat[x],nat[z]],nat[y]],  natadd[natquot[nat[x],nat[y]], natquot[nat[z],nat[y]]]]]

       Out[q13]= True

       Execution time = 0 Seconds
```

Quaife's theorem **(Q14)** is not valid without the additional hypothesis that **y** be nonzero.

```
In[9]:= example[q14, "or[member[natadd[nat[x],nat[
           z]], natadd[nat[y],natmul[nat[y],natquot[nat[x],nat[y]]],
         natmul[nat[y],natquot[nat[z],nat[y]]]]],
         equal[natadd[natquot[nat[x],nat[y]],natquot[nat[z],nat[y]],set[0]],
         natquot[natadd[nat[x],nat[z]],nat[y]]]]//not//not", ""]

In[q14]:=
 or[member[natadd[nat[x],nat[z]], natadd[nat[y],natmul[nat[y],natquot[nat[x],nat[y]]],  natmul[
   nat[y],natquot[nat[z],nat[y]]]]],  equal[natadd[natquot[nat[x],nat[y]],natquot[nat[z],nat[
   y]],set[0]],  natquot[natadd[nat[x],nat[z]],nat[y]]]]//not//not

Out[q14]= not[equal[0, nat[y]]]

Execution time = 0 Seconds
```

When **y = 0**, the first literal of Quaife's clause **(Q14)** is false because it reduces to the statement that **x + z < 0**. The second literal is also false when **y = 0** because it reduces to the statement **1 = 0**.