Quine’s definition of ordinal numbers

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```
In[1]:= SetDirectory["l:"]; << goedel.10may26a;<< tools.m

        :Package Title: goedel.10may26a              2010 May 26 at 11:45 a.m.

        It is now:  2010 May 26 at 12:0

        Loading Simplification Rules

        TOOLS.M                                     Revised 2010 February 26

        weightlimit = 40
```

summary

In this notebook, Quine’s characterization of the class of ordinal numbers is verified. See page 157 of the following reference. It should be noted that Quine does not assume the axiom of regularity, and neither does the **GOEDEL** program.

Reference:

```
In[2]:= "Willard van Orman Quine, Set Theory and its Logic, The Belknap Press of Harvard
          University Press, Cambridge, Mass., 1963.  Third edition (paperback), 1971.";
```

Quine’s characterization involves the class of relations obtained by intersecting well-orderingswith the diversity relation **Di**. This class is the class of well-founded connex relations.

```
In[37]:= image[IMAGE[id[Di]], WO]

Out[37]= intersection[CONNEX, WF]
```

It should be noted that connex well-founded relations are automatically irreflexive and transitive.

```
In[38]:= subclass[intersection[CONNEX, WF], intersection[TRV, P[Di]]]

Out[38]= True
```

derivation

Theorem. If **x** is full and if **id[x]** ∘ **E** is transitive, then every member of **x** is full.

```
In[3]:= SubstTest[implies, and[equal[t, u], subclass[composite[t, v], w]],
          subclass[composite[u, v], w], {t → composite[id[x], E, id[x]],
           u → composite[id[x], E], v → E, w → composite[id[x], E]}] // Reverse

Out[3]= or[not[subclass[U[x], x]],
          not[TRANSITIVE[composite[id[x], E]]], subclass[x, FULL]] == True

In[4]:= or[not[subclass[U[x_], x_]],
          not[TRANSITIVE[composite[id[x_], E]]], subclass[x_, FULL]] := True
```

The class of all **x** such that **id[x]** ∘ **E** is transitive is **cliques[S ⋃ complement[E]]**.

```
In[5]:= member[x, cliques[union[S, complement[E]]]]

Out[5]= and[member[x, V], TRANSITIVE[composite[id[x], E]]]
```

Lemma. Temporary rewrite rule obtained by eliminating the variable **x**.

```
In[6]:= Map[equal[V, #] &,
         dif[intersection[FULL, cliques[union[S, complement[E]]]], P[FULL]] // complement//
          Normality]

Out[6]= subclass[U[intersection[FULL, cliques[union[S, complement[E]]]]], FULL] == True

In[7]:= % /. Equal → SetDelayed
```

Lemma.

```
In[8]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
          {u → H[FULL], v → P[FULL], w -> cliques[union[S, complement[E]]]}] // Reverse

Out[8]= subclass[H[FULL], cliques[union[S, complement[E]]]] == True

In[9]:= subclass[H[FULL], cliques[union[S, complement[E]]]] := True
```

Theorem. If a set **x** is full and if **id[x]** ∘ **E** is transitive, then **x** is hereditarily full.

```
In[10]:= SubstTest[and, subclass[u, v], subclass[v, u],
          {u -> intersection[FULL, cliques[union[S, complement[E]]]], v -> H[FULL]}]

Out[10]= equal[H[FULL], intersection[FULL, cliques[union[S, complement[E]]]]] == True

In[11]:= intersection[FULL, cliques[union[S, complement[E]]]] := H[FULL]
```

Theorem. A formula for the class of **x** such that **id[x]** ∘ **E** is well-founded and transitive.

```
In[12]:= SubstTest[intersection, image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], u],
          image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], v], {u → TRV, v → WF}]

Out[12]= image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], intersection[TRV, WF]] ==
          intersection[FUND, cliques[union[S, complement[E]]]]

In[13]:= image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], intersection[TRV, WF]] :=
          intersection[FUND, cliques[union[S, complement[E]]]]
```

It follows that the class of ordinals is the class of full sets **x** for which **id[x]** ∘ **E** is well-founded and transitive.

```
In[14]:= intersection[FULL, intersection[FUND, cliques[union[S, complement[E]]]]]

Out[14]= OMEGA
```

Quine's characterization of the class of ordinals

Since any well-founded connex relation is transitive, one can replace transitive with connex in the characterization of the class of ordinal numbers derived in the preceding section.

Lemma

```
In[17]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
           {u → cartsq[union[intersection[complement[set[0]], ord[x]], U[ord[x]]]],
            v → cartsq[ord[x]], w ->
             union[Id, composite[id[ord[x]], E], composite[inverse[E], id[ord[x]]]]}] // Reverse

Out[17]= subclass[cart[union[intersection[complement[set[0]], ord[x]], U[ord[x]]],
            union[intersection[complement[set[0]], ord[x]], U[ord[x]]]],
          union[Id, composite[id[ord[x]], E], composite[inverse[E], id[ord[x]]]]] == True

In[18]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. If **x** ∈ **Ω**, then **id[x]** ∘ **E** is connex.

```
In[19]:= SubstTest[implies, equal[x, ord[t]], member[x,
            image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]], t → x] // Reverse

Out[19]= or[not[member[x, OMEGA]],
          subclass[cart[union[intersection[x, complement[set[0]]], U[x]],
            union[intersection[x, complement[set[0]]], U[x]]],
           union[Id, composite[id[x], E], composite[inverse[E], id[x]]]]] == True

In[20]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. Variable-free restatement of the preceding lemma.

```
In[21]:= Map[equal[V, #] &, SubstTest[class, x, implies[member[x, u], member[x, v]],
           {u → OMEGA, v -> image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]}]]

Out[21]= subclass[image[IMAGE[composite[id[E], inverse[SECOND]]], OMEGA], CONNEX] == True

In[22]:= subclass[image[IMAGE[composite[id[E], inverse[SECOND]]], OMEGA], CONNEX] := True
```

Corollary. A simplification rule.

```
In[23]:= equal[intersection[OMEGA,
           image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]], OMEGA]

Out[23]= True
```

```
In[24]:= intersection[OMEGA,
           image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]] := OMEGA
```

Theorem. A formula for the class of all sets **x** such that **id[x]** ∘ **E** is well-founded and connex.

```
In[26]:= SubstTest[intersection, image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], u],
          image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], v], {u → WF, v → CONNEX}]

Out[26]= image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], intersection[CONNEX, WF]] ==
          intersection[FUND, image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]]
```

```
In[27]:= image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], intersection[CONNEX, WF]] :=
          intersection[FUND, image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]]
```

Theorem. Quine’s characterization of ordinals as full sets **x** for which **id[x]** ∘ **E** is well-founded and connex.

```
In[28]:= Map[intersection[FULL, #] &,
          SubstTest[intersection, image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], u],
           image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], v],
           {u → intersection[TRV, WF], v → CONNEX}]]

Out[28]= intersection[FULL, REGULAR,
           image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]] == OMEGA
```

```
In[29]:= intersection[FULL, REGULAR,
           image[inverse[IMAGE[composite[id[E], inverse[SECOND]]]], CONNEX]] := OMEGA
```

some related results

In this section some additional facts discovered in the course of this study are derived.

Theorem. A simplification rule.

```
In[31]:= ImageComp[IMAGE[composite[id[E], inverse[SECOND]]],
           inverse[IMAGE[composite[id[E], inverse[SECOND]]]], TRV] // Reverse

Out[31]= image[IMAGE[composite[id[E], inverse[SECOND]]], cliques[union[S, complement[E]]]] ==
          intersection[TRV, image[INVERSE, RS[inverse[E]]]]
```

```
In[32]:= image[IMAGE[composite[id[E], inverse[SECOND]]], cliques[union[S, complement[E]]]] :=
          intersection[TRV, image[INVERSE, RS[inverse[E]]]]
```

Theorem. A simplification rule.

```
In[35]:= ImageComp[IMAGE[composite[id[E], inverse[SECOND]]],
           inverse[IMAGE[composite[id[E], inverse[SECOND]]]], WF] // Reverse

Out[35]= image[IMAGE[composite[id[E], inverse[SECOND]]], FUND] ==
          intersection[WF, image[INVERSE, RS[inverse[E]]]]
```

```
In[36]:= image[IMAGE[composite[id[E], inverse[SECOND]]], FUND] :=
          intersection[WF, image[INVERSE, RS[inverse[E]]]]
```

Observation. If **x** is an ordinal, then **id[x]** ∘ **E** is transitive.

```
In[39]:= TRANSITIVE[composite[id[ord[x]], E]]

Out[39]= True
```

Theorem. A variable-free restatement of this observation.

```
In[33]:= Map[or[subclass[image[IMAGE[composite[id[E], inverse[SECOND]]], OMEGA], TRV], #] &,
          SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
            {t -> IMAGE[composite[id[E], inverse[SECOND]]], u → OMEGA,
             v → cliques[union[S, complement[E]]]}] // Reverse]

Out[33]= subclass[image[IMAGE[composite[id[E], inverse[SECOND]]], OMEGA], TRV] == True

In[34]:= subclass[image[IMAGE[composite[id[E], inverse[SECOND]]], OMEGA], TRV] := True
```