rank function for PRIMES

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```
In[1]:= SetDirectory["l:"]; << goedel.11aug13a

:Package Title: goedel.11aug13a               2011 August 13 at 5:55 p.m.

Loading takes about eleven minutes, half that time due to builtin pauses.

It is now:  2011 Aug 14 at 22:10

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2011 Aug 14 at 22:22
```

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summary

It is shown in this notebook that **inverse[PRIMESEQ]** is the rank function for the set of primes, well ordered by inclusion.

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derivation

Theorem. A rewrite rule for the symmetric restriction of the proper subset relation to the set of primes.

```
In[2]:= AssInt[PS, cartsq[omega], cartsq[PRIMES]]

Out[2]= composite[id[PRIMES], PS, id[PRIMES]] ==
         composite[id[PRIMES], inverse[IMAGE[id[PRIMES]]], E]

In[3]:= composite[id[PRIMES], PS, id[PRIMES]] :=
         composite[id[PRIMES], inverse[IMAGE[id[PRIMES]]], E]
```

Theorem. Any restriction of **id[PRIMES] ∘ E** is well-founded.

```
In[4]:= SubstTest[implies, and[subclass[u, v], WELLFOUNDED[v]], WELLFOUNDED[u],
          {u → composite[id[PRIMES], E, id[x]], v -> composite[id[PRIMES], E]}] // Reverse

Out[4]= WELLFOUNDED[composite[id[PRIMES], inverse[IMAGE[id[x]]], E]] == True

In[5]:= WELLFOUNDED[composite[id[PRIMES], inverse[IMAGE[id[x_]]], E]] := True
```

Theorem. A rewrite rule for the symmetric restriction of **inverse[PS]** to **PRIMES**.

```
In[6]:= composite[id[PRIMES], inverse[PS], id[PRIMES]] // DoubleInverse

Out[6]= composite[id[PRIMES], inverse[PS], id[PRIMES]] ==
         composite[inverse[E], IMAGE[id[PRIMES]], id[PRIMES]]

In[7]:= composite[id[PRIMES], inverse[PS], id[PRIMES]] :=
         composite[inverse[E], IMAGE[id[PRIMES]], id[PRIMES]]
```

Lemma. A simplification rule.

```
In[8]:= Assoc[TC, id[omega], inverse[PRIMESEQ]]

Out[8]= composite[TC, inverse[PRIMESEQ]] == inverse[PRIMESEQ]

In[9]:= composite[TC, inverse[PRIMESEQ]] := inverse[PRIMESEQ]
```

The main theorem is obtained as an application of the uniqueness theorem for rank functions (see the posted notebook **wf-rec-7.nb**), and the following variable-free restatement of the recursion relation for **inverse[PRIMESEQ]** derived in the posted notebook **inprmseq.nb**.

```
In[10]:= composite[IMAGE[inverse[PRIMESEQ]], id[PRIMES]]

Out[10]= inverse[PRIMESEQ]
```

Main Theorem. The function **inverse[PRIMESEQ]** is the rank function for the set of primes.

```
In[11]:= SubstTest[implies, and[thin[y], WELLFOUNDED[inverse[y]],
            equal[w, composite[TC, IMAGE[w], VERTSECT[y]]]]],
           equal[w, rec[composite[TC, IMAGE[SECOND], SECOND], y]],
           {w → union[cart[complement[PRIMES], set[0]], inverse[PRIMESEQ]],
            y → composite[inverse[E], IMAGE[id[PRIMES]], id[PRIMES]]}] // Reverse

Out[11]= equal[rec[composite[TC, IMAGE[SECOND], SECOND],
            composite[inverse[E], IMAGE[id[PRIMES]], id[PRIMES]]],
           union[cart[complement[PRIMES], set[0]], inverse[PRIMESEQ]]] == True

In[12]:= rec[composite[TC, IMAGE[SECOND], SECOND],
           composite[inverse[E], IMAGE[id[PRIMES]], id[PRIMES]]] :=
          union[cart[complement[PRIMES], set[0]], inverse[PRIMESEQ]]
```