multiplicative group of nonzero rational numbers

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```
In[1]:= SetDirectory["l:"]; << goedel.12sep24a

:Package Title: goedel.12sep24a            2012 September 24 at 5:55 p.m.

Loading takes about sixteen minutes, half that time due to builtin pauses.

It is now:  2012 Sep 25 at 15:23

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2012 Sep 25 at 15:38
```

summary

The nonzero rationals form a group under multiplication. The derivation of this fact presented in this noteboook makes some elementary use of category theory as well as the theory of monoids in order to avoid having to introduce variables for individual rational numbers.

derivation

The following abbreviations for the rational numbers zero and one will be used below.

```
In[2]:= zero := cart[Z, set[id[omega]]]

In[3]:= one := id[Z]
```

Theorem. Rational multiplication is a category.

```
In[4]:= SubstTest[category, monoid[x], x → RATMUL] // Reverse

Out[4]= category[RATMUL] == True

In[5]:= category[RATMUL] := True
```

Theorem. A simplification rule.

```
In[6]:= Assoc[RATMUL, id[cart[RATS, V]], id[cart[x, y]]]

Out[6]= composite[RATMUL, id[cart[intersection[RATS, x], y]]] ==
         composite[RATMUL, id[cart[x, y]]]

In[7]:= composite[RATMUL, id[cart[intersection[RATS, x_], y_]]] :=
         composite[RATMUL, id[cart[x, y]]]
```

Theorem. A simplification rule.

```
In[8]:= Assoc[RATMUL, id[cart[V, RATS]], id[cart[x, y]]]

Out[8]= composite[RATMUL, id[cart[x, intersection[RATS, y]]]] ==
         composite[RATMUL, id[cart[x, y]]]

In[9]:= composite[RATMUL, id[cart[x_, intersection[RATS, y_]]]] :=
         composite[RATMUL, id[cart[x, y]]]
```

Theorem. The invertible elements of a category form a subcategory.

```
In[10]:= SubstTest[category,
           composite[cat[x], id[cartsq[domain[inv[cat[x]]]]]], x → RATMUL] // Reverse

Out[10]= category[composite[RATMUL, id[cart[complement[set[cart[Z, set[id[omega]]]]],
              complement[set[cart[Z, set[id[omega]]]]]]]]] == True

In[11]:= % /. Equal → SetDelayed
```

Corollary. The set of non-zero rational numbers is binary closed under multiplication.

```
In[12]:= SubstTest[image, cat[x], cartsq[domain[inv[cat[x]]]], x → RATMUL] // Reverse

Out[12]= image[RATMUL, cart[complement[set[cart[Z, set[id[omega]]]]],
            complement[set[cart[Z, set[id[omega]]]]]]] ==
          intersection[RATS, complement[set[cart[Z, set[id[omega]]]]]]

In[13]:= image[RATMUL, cart[complement[set[cart[Z, set[id[omega]]]]],
            complement[set[cart[Z, set[id[omega]]]]]]] :=
          intersection[RATS, complement[set[cart[Z, set[id[omega]]]]]]
```

Lemma. The product of zero and any rational number is zero.

```
In[14]:= SubstTest[implies, equal[x, rat[t]],
           member[x, image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
             set[cart[Z, set[id[omega]]]]]], t → x] // Reverse

Out[14]= or[equal[cart[Z, set[id[omega]]], ratmul[cart[Z, set[id[omega]]], x]],
           not[member[x, RATS]]] == True

In[15]:= or[equal[cart[Z, set[id[omega]]], ratmul[cart[Z, set[id[omega]]], x_]],
           not[member[x_, RATS]]] := True
```

Theorem. An inclusion. (This will shortly be strengthened to an equation.)

```
In[16]:= Map[equal[V, #] &,
          dif[RATS, image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
              set[cart[Z, set[id[omega]]]]]] // complement // Normality]

Out[16]= subclass[RATS, image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
            set[cart[Z, set[id[omega]]]]]] == True

In[17]:= % /. Equal → SetDelayed
```

Theorem. A general inclusion.

```
In[18]:= SubstTest[subclass, intersection[u, v], u,
           {u → RATS, v -> image[image[inverse[RATMUL], x], y]}] // Reverse

Out[18]= subclass[image[image[inverse[RATMUL], x], y], RATS] == True

In[19]:= subclass[image[image[inverse[RATMUL], x_], y_], RATS] := True
```

Theorem. A temporary simplification rule.

```
In[20]:= SubstTest[and, subclass[u, v], subclass[v, u],
          {u → RATS, v -> image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
             set[cart[Z, set[id[omega]]]]]}]

Out[20]= equal[RATS, image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
            set[cart[Z, set[id[omega]]]]]] == True

In[21]:= image[image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]],
           set[cart[Z, set[id[omega]]]]] := RATS
```

Theorem. A simplification rule for the restriction of rational multiplication to non-zero rational numbers.

```
In[22]:= equal[composite[id[complement[set[zero]]], RATMUL],
           composite[RATMUL, id[cart[complement[set[cart[Z, set[id[omega]]]]],
              complement[set[cart[Z, set[id[omega]]]]]]]]] // AssertTest

Out[22]= equal[composite[RATMUL, id[cart[complement[set[cart[Z, set[id[omega]]]]],
              complement[set[cart[Z, set[id[omega]]]]]]]],
           composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] == True

In[23]:= composite[RATMUL, id[cart[complement[set[cart[Z, set[id[omega]]]]],
             complement[set[cart[Z, set[id[omega]]]]]]]] :=
          composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]
```

Corollary. A simplification rule.

```
In[27]:= Map[dif[cartsq[RATS], #] &,
          IminComp[RATMUL, id[cart[complement[set[cart[Z, set[id[omega]]]]],
             complement[set[cart[Z, set[id[omega]]]]]]], V]]

Out[27]= image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]] == union[
           cart[RATS, set[cart[Z, set[id[omega]]]]], cart[set[cart[Z, set[id[omega]]]], RATS]]
```

```
In[28]:= image[inverse[RATMUL], set[cart[Z, set[id[omega]]]]] := union[
           cart[RATS, set[cart[Z, set[id[omega]]]]], cart[set[cart[Z, set[id[omega]]]], RATS]]
```

Corollary. If the product of two rational numbers is zero, then at least one of them is zero.

```
In[30]:= SubstTest[member, pair[rat[x], rat[y]],
          image[inverse[RATMUL], t], t -> set[cart[Z, set[id[omega]]]]]

Out[30]= equal[cart[Z, set[id[omega]]], ratmul[rat[x], rat[y]]] ==
          or[equal[cart[Z, set[id[omega]]], rat[x]], equal[cart[Z, set[id[omega]]], rat[y]]]

In[31]:= equal[cart[Z, set[id[omega]]], ratmul[rat[x_], rat[y_]]] :=
          or[equal[cart[Z, set[id[omega]]], rat[x]], equal[cart[Z, set[id[omega]]], rat[y]]]
```

Theorem. The non-zero rational numbers form a monoid under multiplication.

```
In[32]:= SubstTest[implies, and[member[x, MONOIDS], member[e[x], y], member[y, binclosed[x]]],
           member[composite[x, id[cart[y, y]]], MONOIDS], {x → RATMUL,
            y -> intersection[RATS, complement[set[cart[Z, set[id[omega]]]]]]}] // Reverse

Out[32]= member[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL], MONOIDS] == True

In[33]:= member[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL], MONOIDS] :=
          True
```

Theorem. A rewrite rule for the neutral element for the restriction of rational multiplication to nonzero rationals.

```
In[34]:= SubstTest[or, equal[e[x], e[composite[x, id[cart[y, y]]]]], not[member[x, MONOIDS]],
           not[member[e[x], y]], not[subclass[image[x, cart[y, y]], y]],
           {x → RATMUL, y -> complement[set[cart[Z, set[id[omega]]]]]}] // Reverse

Out[34]= equal[e[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]], id[Z]] == True

In[35]:= e[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] := id[Z]
```

Corollary. A rewrite rule for the set of identity elements.

```
In[36]:= SubstTest[ids, monoid[t],
           t -> composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] // Reverse

Out[36]= ids[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] == set[id[Z]]

In[37]:= ids[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] := set[id[Z]]
```

Theorem. A rewrite rule for the inversion function.

```
In[38]:= SubstTest[intersection, image[inverse[t], ids[t]], inverse[image[inverse[t], ids[t]]],
          t -> composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]]

Out[38]= inv[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] ==
          composite[id[RATS], INVERSE, id[RATS]]

In[39]:= inv[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]] :=
          composite[id[RATS], INVERSE, id[RATS]]
```

Main Theorem. The non-zero rational numbers form a group under multiplication.

```
In[40]:= SubstTest[and, member[x, MONOIDS], equal[domain[inv[x]], range[x]],
          x -> composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL]]

Out[40]= member[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL], GROUPS] == True

In[41]:= member[composite[id[complement[set[cart[Z, set[id[omega]]]]]], RATMUL], GROUPS] := True
```