semigroup complexes

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In[1]:= **SetDirectory["l:"]; << goedel.11jun28a**

```
:Package Title: goedel.11jun28a                2011 June 28 at 7:30 p.m.

Loading takes about eleven minutes, half that time due to builtin pauses.

It is now:  2011 Jun 29 at 8:53

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now:  2011 Jun 29 at 9:4
```

summary

A **complex** for a semigroup **semigp[x]** is a subset of **fix[domain[semigp[x]]]**. The multiplication of complexes provides an example of a new semigroup constructed from an old one. The multiplication of complexes of a group forms a monoid.

derivation

In general, one can construct an associative function from any thin associative relation:

In[8]:= **associative[composite[IMAGE[assoc[thinpart[x]]], CART]]**

Out[8]= True

This can be specialized to the case of a semigroup.

Theorem.

In[9]:= **SubstTest[associative,
composite[IMAGE[assoc[thinpart[t]]], CART], t → semigp[x]] // Reverse**

Out[9]= associative[composite[IMAGE[semigp[x]], CART]] == True

In[10]:= **associative[composite[IMAGE[semigp[x_]], CART]] := True**

In general, this yields a proper class, not a set, so it cannot belong to **SEMIGPS**.

Theorem.

```
In[12]:= member[composite[IMAGE[semigp[x]], CART], V] // AssertTest

Out[12]= member[composite[IMAGE[semigp[x]], CART], V] == False

In[13]:= member[composite[IMAGE[semigp[x_]], CART], V] := False
```

To obtain a semigroup, one needs to restrict this associative function.

Lemma.

```
In[18]:= SubstTest[implies, and[associative[t], FUNCTION[t], subclass[image[t, cart[y, y]], y]],
           associative[composite[t, id[cart[y, y]]]],
           t -> composite[IMAGE[semigp[x]], CART]] // Reverse

Out[18]= or[associative[composite[IMAGE[semigp[x]], CART, id[cart[y, y]]]],
           not[subclass[image[IMAGE[semigp[x]], image[CART, cart[y, y]]], y]]] == True

In[19]:= or[associative[composite[IMAGE[semigp[x_]], CART, id[cart[y_, y_]]]],
           not[subclass[image[IMAGE[semigp[x]], image[CART, cart[y_, y_]]], y_]]] := True
```

Theorem. The multiplication of complexes of a semigroup is associative.

```
In[21]:= SubstTest[implies, subclass[image[IMAGE[semigp[x]], image[CART, cart[y, y]]], y],
           associative[composite[IMAGE[semigp[x]], CART, id[cart[y, y]]]],
           y → P[fix[domain[semigp[x]]]]] // Reverse

Out[21]= associative[composite[IMAGE[semigp[x]], CART,
            id[cart[P[fix[domain[semigp[x]]]], P[fix[domain[semigp[x]]]]]]]] == True

In[22]:= associative[composite[IMAGE[semigp[x_]], CART,
            id[cart[P[fix[domain[semigp[x_]]]], P[fix[domain[semigp[x_]]]]]]]] := True
```

Lemma.

```
In[31]:= member[composite[IMAGE[binop[x]], CART,
            id[cart[P[fix[domain[binop[x]]]], P[fix[domain[binop[x]]]]]]],
           map[cart[P[fix[domain[binop[x]]]], P[fix[domain[binop[x]]]]],
            P[fix[domain[binop[x]]]]]] // AssertTest

Out[31]= member[composite[IMAGE[binop[x]], CART,
            id[cart[P[fix[domain[binop[x]]]], P[fix[domain[binop[x]]]]]]],
           map[cart[P[fix[domain[binop[x]]]], P[fix[domain[binop[x]]]]],
            P[fix[domain[binop[x]]]]]] == True

In[32]:= (% /. x → x_) /. Equal → SetDelayed
```

Theorem. The multiplication of complexes for any binary operation is a binary operation.

```
In[33]:= SubstTest[member, t, map[cartsq[fix[domain[t]]], fix[domain[t]]],
           t -> composite[IMAGE[binop[x]], CART, id[cart[y, y]]]] /. y → P[fix[domain[binop[x]]]]

Out[33]= member[composite[IMAGE[binop[x]], CART,
            id[cart[P[fix[domain[binop[x]]]], P[fix[domain[binop[x]]]]]]], BINOPS] == True

In[34]:= member[composite[IMAGE[binop[x_]], CART,
            id[cart[P[fix[domain[binop[x_]]]], P[fix[domain[binop[x_]]]]]]], BINOPS] := True
```

Corollary. Specialization to semigroups.

```
In[36]:= SubstTest[member, composite[IMAGE[binop[t]], CART,
            id[cart[P[fix[domain[binop[t]]]], P[fix[domain[binop[t]]]]]]],
           BINOPS, t → semigp[x]] // Reverse

Out[36]= member[composite[IMAGE[semigp[x]], CART,
            id[cart[P[fix[domain[semigp[x]]]], P[fix[domain[semigp[x]]]]]]], BINOPS] == True

In[37]:= member[composite[IMAGE[semigp[x_]], CART,
            id[cart[P[fix[domain[semigp[x_]]]], P[fix[domain[semigp[x_]]]]]]], BINOPS] := True
```

Theorem. The multiplication of complexes of a semigroup is a semigroup.

```
In[38]:= member[composite[IMAGE[semigp[x]], CART,
            id[cart[P[fix[domain[semigp[x]]]], P[fix[domain[semigp[x]]]]]]],
           SEMIGPS] // AssertTest

Out[38]= member[composite[IMAGE[semigp[x]], CART,
            id[cart[P[fix[domain[semigp[x]]]], P[fix[domain[semigp[x]]]]]]], SEMIGPS] == True

In[39]:= member[composite[IMAGE[semigp[x_]], CART,
            id[cart[P[fix[domain[semigp[x_]]]], P[fix[domain[semigp[x_]]]]]]], SEMIGPS] := True
```

Comment. The result without wrappers is already available.

```
In[41]:= implies[member[x, SEMIGPS], member[
           composite[IMAGE[x], CART, id[cart[P[fix[domain[x]]], P[fix[domain[x]]]]]], SEMIGPS]]

Out[41]= True
```

categories

The case of a category is considered in this section.

Theorem. An associative function constructed from any category.

```
In[44]:= SubstTest[associative,
           composite[IMAGE[assoc[thinpart[t]]], CART], t → cat[x]] // Reverse

Out[44]= associative[composite[IMAGE[cat[x]], CART]] == True
```

```
In[45]:= associative[composite[IMAGE[cat[x_]], CART]] := True
```

Theorem. The above function is never a set.

```
In[46]:= member[composite[IMAGE[cat[x]], CART], V] // AssertTest

Out[46]= member[composite[IMAGE[cat[x]], CART], V] == False

In[47]:= member[composite[IMAGE[cat[x_]], CART], V] := False
```

Theorem. The multiplication of complexes in a category is associative.

```
In[50]:= (SubstTest[implies, and[associative[t], FUNCTION[t],
           subclass[image[t, cart[y, y]], y]], associative[composite[t, id[cart[y, y]]]]],
         t -> composite[IMAGE[cat[x]], CART]] // Reverse) /. y → P[range[cat[x]]]

Out[50]= associative[
          composite[IMAGE[cat[x]], CART, id[cart[P[range[cat[x]]], P[range[cat[x]]]]]]] == True

In[51]:= associative[composite[IMAGE[cat[x_]], CART,
           id[cart[P[range[cat[x_]]], P[range[cat[x_]]]]]]] := True
```

complexes in a group

Lemma.

```
In[91]:= Map[image[#, cart[V, y]] &,
          composite[gp[x], id[cart[set[e[gp[x]]], V]]]] // FastReifTriNormality

Out[91]= image[gp[x], cart[set[e[gp[x]]], y]] == intersection[y, range[gp[x]]]

In[92]:= image[gp[x_], cart[set[e[gp[x_]]], y_]] := intersection[y, range[gp[x]]]
```

Lemma. A simplification rule.

```
In[86]:= image[V, set[e[gp[x]]]] // Normality

Out[86]= image[V, set[e[gp[x]]]] == image[V, gp[x]]

In[87]:= image[V, set[e[gp[x_]]]] := image[V, gp[x]]
```

Theorem. A simplification rule.

```
In[96]:= composite[IMAGE[gp[x]], CART, LEFT[set[e[gp[x]]]]] // FastReifNormality

Out[96]= composite[IMAGE[gp[x]], IMAGE[LEFT[e[gp[x]]]]] == IMAGE[id[range[gp[x]]]]

In[97]:= composite[IMAGE[gp[x_]], IMAGE[LEFT[e[gp[x_]]]]] := IMAGE[id[range[gp[x]]]]
```

Dual Theorem.

In[115]:=
```
SubstTest[composite, IMAGE[gp[t]], IMAGE[LEFT[e[gp[t]]]], t → flip[gp[x]]] // Reverse
```

Out[115]=
```
composite[IMAGE[gp[x]], IMAGE[RIGHT[e[gp[x]]]]] == IMAGE[id[range[gp[x]]]]
```

In[116]:=
```
composite[IMAGE[gp[x_]], IMAGE[RIGHT[e[gp[x_]]]]] := IMAGE[id[range[gp[x]]]]
```

Theorem. The singleton of the identity element of a group is an identity element for multiplication of complexes.

In[118]:=
```
SubstTest[and, member[t, fix[domain[u]]],
  subclass[composite[u, LEFT[t]], Id], subclass[composite[u, RIGHT[t]], Id],
  {t → set[e[gp[x]]], u -> composite[IMAGE[gp[x]], CART, id[cartsq[P[range[gp[x]]]]]]}]
```

Out[118]=
```
member[set[e[gp[x]]],
  ids[composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]]]] == True
```

In[119]:=
```
member[set[e[gp[x_]]], ids[composite[IMAGE[gp[x_]],
    CART, id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]]]] := True
```

Theorem. If **y** is an identity element for a binary operation **x**, then **ids[x] = set[y]**.

In[123]:=
```
SubstTest[implies, equal[x, binop[t]], implies[
   and[member[x, BINOPS], member[y, ids[x]]], equal[ids[x], set[y]]], t → x] // Reverse
```

Out[123]=
```
or[equal[ids[x], set[y]], not[member[x, BINOPS]], not[member[y, ids[x]]]] == True
```

In[124]:=
```
or[equal[ids[x_], set[y_]], not[member[x_, BINOPS]], not[member[y_, ids[x_]]]] := True
```

Theorem. The multiplication of complexes in a group is a binary operation.

In[127]:=
```
SubstTest[member, composite[IMAGE[semigp[t]], CART,
   id[cart[P[fix[domain[semigp[t]]]], P[fix[domain[semigp[t]]]]]]],
  BINOPS, t → gp[x]] // Reverse
```

Out[127]=
```
member[composite[IMAGE[gp[x]], CART,
   id[cart[P[range[gp[x]]], P[range[gp[x]]]]]], BINOPS] == True
```

In[128]:=
```
member[composite[IMAGE[gp[x_]], CART,
   id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]], BINOPS] := True
```

Theorem. A formula for the set of identity elements for multiplication of complexes of a group.

```
In[129]:=
SubstTest[or, equal[ids[t], set[y]], not[member[t, BINOPS]], not[member[y, ids[t]]],
  {t → composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]],
   y → set[e[gp[x]]]}] // Reverse

Out[129]=
equal[ids[composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]]],
  set[set[e[gp[x]]]]] == True

In[131]:=
ids[composite[IMAGE[gp[x_]], CART, id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]]] :=
 set[set[e[gp[x]]]]
```

Corollary. A formula for the identity element for multiplication of complexes of a group.

```
In[132]:=
SubstTest[A, ids[t],
 t -> composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]]]

Out[132]=
e[composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]]] ==
 set[e[gp[x]]]

In[133]:=
e[composite[IMAGE[gp[x_]], CART, id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]]] :=
 set[e[gp[x]]]
```

Theorem. The multiplication of complexes in a group is associative.

```
In[137]:=
SubstTest[associative, composite[IMAGE[semigp[t]], CART,
   id[cart[P[fix[domain[semigp[t]]]], P[fix[domain[semigp[t]]]]]]], t → gp[x]] // Reverse

Out[137]=
associative[
  composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]]] == True

In[138]:=
associative[
  composite[IMAGE[gp[x_]], CART, id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]]] := True
```

Corollary. The multiplication of complexes in a group is a semigroup.

```
In[140]:=
member[composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]],
  SEMIGPS] // AssertTest

Out[140]=
member[composite[IMAGE[gp[x]], CART,
   id[cart[P[range[gp[x]]], P[range[gp[x]]]]]], SEMIGPS] == True

In[141]:=
member[composite[IMAGE[gp[x_]], CART,
   id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]], SEMIGPS] := True
```

A better result is the following.

Theorem. The multiplication of complexes of a group is a monoid.

In[142]:=

```
member[composite[IMAGE[gp[x]], CART, id[cart[P[range[gp[x]]], P[range[gp[x]]]]]],
  MONOIDS] // AssertTest
```

Out[142]=

```
member[composite[IMAGE[gp[x]], CART,
   id[cart[P[range[gp[x]]], P[range[gp[x]]]]]], MONOIDS] == True
```

In[143]:=

```
member[composite[IMAGE[gp[x_]], CART,
   id[cart[P[range[gp[x_]]], P[range[gp[x_]]]]]], MONOIDS] := True
```