successors of natural numbers

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In[1]:=
```
SetDirectory["l:"]; << goedel.09oct09a; << tools.m
```

```
:Package Title: goedel.09oct09a            2009 October 9 at 3:50 p.m.

It is now:  2009 Oct 13 at 11:55

Loading Simplification Rules

TOOLS.M                                    Revised 2009 September 15

weightlimit = 40
```

summary

This notebook is concerned various characterizations of the successor relation for natural numbers found in the literature. For example, the successor relation for natural numbers can be viewed as the cover relation corresponding to the membership relation on natural numbers. This fact is recognized by the current rules in the **GOEDEL** program.

In[2]:=
```
cover[composite[id[omega], E]]
```

Out[2]=
```
composite[id[omega], SUCC]
```

In the early literature on well-ordering, a general concept of **immediate successor** was introduced. See for example:

In[3]:=
```
"Cesare Burali-Forti, A question on transfinite numbers, Rendiconti del
 Circolo matematico di Palermo (1897), vol. 8, pp. 154-164.  Reprinted on
 pp. 104-111 in From Frege to Gödel: A Sourcebook in Mathematical Logic,
 1879-1931, edited by Jean van Heijenoort, 1967. (ISBN 0-674-32450)";
```

The concept of immediate successor for well-orders can be formulated in several equivalent ways. New rewrite rules are needed for the **GOEDEL** program to recognize some of these. In this notebook a description given by Patrick Suppes is considered.

In[4]:=
```
"Patrick Suppes, Axiomatic Set Theory, Dover Publications,
 Inc., New York, 1972. See page 76. (ISBN 0-486-61630-4)";
```

a wrapper-free characterization with variables

The following characterization of successor for natural numbers is obtained by removing the **nat** wrapper from an existing rewrite rule in the **GOEDEL** program.

Theorem. A characterization of the successor of a natural number.

```
In[5]:= SubstTest[implies, equal[y, nat[t]],
          or[equal[y, succ[x]], member[x, U[y]], not[member[x, y]]], t → y] // Reverse

Out[5]= or[equal[y, succ[x]], member[x, U[y]], not[member[x, y]], not[member[y, omega]]] == True

In[6]:= or[equal[y_, succ[x_]], member[x_, U[y_]],
          not[member[x_, y_]], not[member[y_, omega]]] := True
```

cover[E] rules

Although the **GOEDEL** program already recognizes the successor relation for natural numbers as the cover relation for the membership relation on natural numbers, some related facts are not yet recognized. For instance, the successor relation on **omega** is a restriction of the cover relation of the global membership relation.

Theorem.

```
In[7]:= SubstTest[intersection, Di, x,
          complement[composite[intersection[Di, x], intersection[Di, x]]],
          x → composite[id[omega], E]] // Reverse

Out[7]= composite[id[omega], cover[E]] == composite[id[omega], SUCC]

In[8]:= composite[id[omega], cover[E]] := composite[id[omega], SUCC]
```

A similar rule holds for ordinal numbers. One need only replace **omega** with **OMEGA**.

Theorem.

```
In[9]:= SubstTest[intersection, Di, x,
          complement[composite[intersection[Di, x], intersection[Di, x]]],
          x → composite[id[OMEGA], E]] // Reverse

Out[9]= composite[id[OMEGA], cover[E]] == composite[id[OMEGA], SUCC]

In[10]:= composite[id[OMEGA], cover[E]] := composite[id[OMEGA], SUCC]
```

The **GOEDEL** program does not automatically assume the axiom of regularity. In the absence of the axiom of regularity, the membership relation need not be irreflexive. When restricted to the natural numbers, however, membership is irreflexive. The following rewrite rule expresses this fact.

Theorem.

```
In[11]:= AssInt[Di, restrict[S, omega, omega], restrict[E, omega, omega]]

Out[11]= composite[id[omega], intersection[Di, E]] == composite[id[omega], E]

In[12]:= composite[id[omega], intersection[Di, E]] := composite[id[omega], E]
```

PS rules

The membership relation **E** and the proper-subset relation **PS** coincide when restricted to the set **omega** of natural numbers. The cover relation for **PS** restricted to natural numbers is another characterization of the successor relation that is recognized by the **GOEDEL** program.

```
In[13]:= restrict[cover[PS], omega, omega]

Out[13]= composite[id[omega], SUCC]
```

When expressions involving the restriction of **PS** to **omega** are encountered, the **GOEDEL** program automatically tries to convert them to corresponding expressions involving the membership relation **E**. Mixed expressions involving both **PS** and **E** may be encountered. The following rewrite rule deals with a situation of this kind.

Theorem.

```
In[14]:= AssInt[composite[id[omega], E], S, Di]

Out[14]= composite[id[omega], intersection[E, PS]] == composite[id[omega], E]

In[15]:= composite[id[omega], intersection[E, PS]] := composite[id[omega], E]
```

Corollary.

```
In[16]:= AssInt[composite[id[omega], E], PS, x]

Out[16]= composite[id[omega], intersection[E, PS, x]] == composite[id[omega], intersection[E, x]]

In[17]:= composite[id[omega], intersection[E, PS, x_]] :=
          composite[id[omega], intersection[E, x]]
```

Lemma.

```
In[18]:= SubstTest[composite, restrict[PS, t, t], restrict[PS, t, t], t → omega]

Out[18]= composite[id[omega], inverse[IMAGE[inverse[PS]]], inverse[IMAGE[id[omega]]], E] ==
          composite[id[omega], inverse[BIGCUP], E]

In[19]:= % /. Equal → SetDelayed
```

The following rewrite rule is also needed to derive the results in the next section.

Theorem.

```
In[20]:= Map[intersection[E, #] &, SubstTest[intersection, Di, x,
            complement[composite[intersection[Di, x], intersection[Di, x]]],
            x → composite[id[omega], S, id[omega]]]] // Reverse

Out[20]= composite[id[omega], intersection[E, composite[inverse[BIGCUP], complement[E]]]] ==
          composite[id[omega], SUCC]
```

```
In[21]:= composite[id[omega], intersection[E, composite[inverse[BIGCUP], complement[E]]]] :=
          composite[id[omega], SUCC]
```

Corollary. Another characterization of the successor relation for natural numbers as the cover relation for the restriction of **PS** to **omega**.

```
In[22]:= composite[id[omega], intersection[PS, complement[composite[PS, id[omega], PS]]],
           id[omega]] // DoubleComplement

Out[22]= composite[id[omega], intersection[PS, complement[composite[PS, id[omega], PS]]],
           id[omega]] == composite[id[omega], SUCC]

In[23]:= composite[id[omega], intersection[PS, complement[composite[PS, id[omega], PS]]],
           id[omega]] := composite[id[omega], SUCC]
```

immediate successors

Patrick Suppes defines the concept of **immediate successor** for an arbitrary relation **r** as follows (see page 76, definition 29). An element **y** is an **r**-immediate successor of **x** if **x** is **r**-related to **y** and for all **z**, if **x** is **r**-related to **z**, then either **y = z** or **y** is **r**-related to **z**. The following new rewrite rule is needed for the **GOEDEL** program to recognize this definition of immediate successor for the special case of natural numbers.

Lemma.

```
In[24]:= composite[id[omega], complement[
            composite[complement[inverse[E]], id[omega], SUCC, E]]] // DoubleComplement

Out[24]= composite[id[omega],
           complement[composite[complement[inverse[E]], id[omega], SUCC, E]]] ==
          composite[id[omega], inverse[BIGCUP], complement[E]]

In[25]:= composite[id[omega],
           complement[composite[complement[inverse[E]], id[omega], SUCC, E]]] :=
          composite[id[omega], inverse[BIGCUP], complement[E]]
```

The concept of immediate successor for natural numbers is now recognized by the **GOEDEL** program:

```
In[26]:= class[pair[x, y], and[member[x, y], member[y, omega], forall[z,
            implies[and[member[x, z], member[z, omega]], or[equal[y, z], member[y, z]]]]]]

Out[26]= composite[id[omega], SUCC]
```

When the order of evaluation is changed, additional rewrite rules are needed.

Theorem. A related simplification rule.

```
In[27]:= composite[id[omega], intersection[E, complement[
             composite[complement[inverse[E]], id[omega], SUCC, E]]]] // DoubleComplement

Out[27]= composite[id[omega],
           intersection[E, complement[composite[complement[inverse[E]], id[omega], SUCC, E]]]] ==
          composite[id[omega], SUCC]

In[28]:= composite[id[omega], intersection[E,
            complement[composite[complement[inverse[E]], id[omega], SUCC, E]]]] :=
          composite[id[omega], SUCC]
```

The above rewrite rule is needed when the definition of immediate successor given by Suppes is first formulated for a general relation and then specialized to the case of **composite[id[omega], E]**.

```
In[29]:= class[pair[x, y], and[member[pair[x, y], r],
            forall[z, implies[member[pair[x, z], r], or[equal[y, z], member[pair[y, z], r]]]]]] /.
          r → composite[id[omega], E]

Out[29]= composite[id[omega], SUCC]
```