a characterization of OMEGA

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```
In[1]:= SetDirectory["l:"]; << goedel.08mar28a; << tools.m

        :Package Title: goedel.08mar28a                    2008 March 28 at 5:30 p.m.

        It is now:  2008 Apr 1 at 7:54

        Loading Simplification Rules

        TOOLS.M                                      Revised 2008 February 12

        weightlimit = 40
```

summary

The class **OMEGA** of all ordinal numbers is successor-invariant and closed under unions of chains.

```
In[2]:= Uchains[OMEGA]

Out[2]= OMEGA

In[3]:= invariant[SUCC, OMEGA]

Out[3]= True
```

These two properties characterize the class of ordinals. It is shown in this notebook that the class **OMEGA** is the smallest successor-invariant class closed under unions of chains. This result holds independently of the axiom of regularity.

derivation

Lemma. (Ordinals are chains.)

```
In[4]:= SubstTest[implies, and[member[x, y], subclass[y, z]],
          member[x, z], {y → OMEGA, z → chains[S]}] // Reverse

Out[4]= or[not[member[x, OMEGA]], subclass[cart[x, x], union[S, inverse[S]]]] == True

In[5]:= or[not[member[x_, OMEGA]], subclass[cart[x_, x_], union[S, inverse[S]]]] := True
```

Lemma.

```
In[7]:= Map[not, SubstTest[and, implies[p1, p2], implies[and[p2, p3], p4],
           not[implies[and[p1, p3], p4]], {p1 -> member[x, OMEGA], p2 → member[x, chains[S]],
            p3 → subclass[x, y], p4 → member[U[x], Uchains[y]]}]] // Reverse

Out[7]= or[member[U[x], Uchains[y]], not[member[x, OMEGA]], not[subclass[x, y]]] == True

In[8]:= or[member[U[x_], Uchains[y_]], not[member[x_, OMEGA]], not[subclass[x_, y_]]] := True
```

Lemma. (For convenience, the temporary abbreviation **t** is used here for the least ordinal that does not belong to **x**. This temporary variable is eliminated from the final statement of the lemma.)

```
In[9]:= (Map[not, SubstTest[and, implies[and[p3, p4], p5], implies[p5, p7],
             implies[and[p4, p7], p8], implies[and[p5, p8], p9], not[implies[and[p3, p4], p9]],
             {p3 → not[subclass[OMEGA, x]], p4 → equal[t, A[dif[OMEGA, x]]],
              p5 → member[t, OMEGA], p6 → not[member[t, x]], p7 → subclass[t, OMEGA], p8 →
               subclass[t, x], p9 → member[U[t], Uchains[x]]}]] // Reverse) /. t → A[dif[OMEGA, x]]

Out[9]= or[member[U[A[intersection[OMEGA, complement[x]]]], Uchains[x]], subclass[OMEGA, x]] ==
         True

In[10]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma.

```
In[13]:= SubstTest[implies, equal[Uchains[x], y],
           or[member[U[A[intersection[OMEGA, complement[x]]]], y], subclass[OMEGA, x]],
           y → x] // Reverse

Out[13]= or[member[U[A[intersection[OMEGA, complement[x]]]], x],
           not[equal[x, Uchains[x]]], subclass[OMEGA, x]] == True

In[14]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma.

```
In[16]:= (Map[not, SubstTest[and, implies[and[p3, p4], p5], implies[and[p3, p4], p6],
              implies[and[p1, p3, p4], q1], implies[and[p6, q1], q2],
              not[implies[and[p1, p3, p4], q3]], {p1 -> equal[x, Uchains[x]], p3 →
                not[subclass[OMEGA, x]], p4 → equal[t, A[dif[OMEGA, x]]], p5 → member[t, OMEGA],
               p6 → not[member[t, x]], q1 → member[U[t], x], q2 → not[equal[U[t], t]],
               q3 → equal[t, succ[U[t]]]}]] /. t → A[dif[OMEGA, x]]) // Reverse

Out[16]= or[equal[A[intersection[OMEGA, complement[x]]],
            succ[U[A[intersection[OMEGA, complement[x]]]]]],
           not[equal[x, Uchains[x]]], subclass[OMEGA, x]] == True

In[17]:= (% /. x → x_) /. Equal → SetDelayed
```

Main Theorem. The class of ordinals is the least class that is successor-invariant and closed under unions of chains.

```
In[18]:= Map[not, SubstTest[and, implies[and[p1, p3], q3], implies[and[p2, q1], q4],
            not[implies[and[p1, p2], not[p3]]], {p1 → equal[x, Uchains[x]],
             p2 → subclass[image[SUCC, x], x], p3 → not[subclass[OMEGA, x]],
             q1 → member[U[A[intersection[OMEGA, complement[x]]]], x],
             q3 → equal[A[intersection[OMEGA, complement[x]]],
               succ[U[A[intersection[OMEGA, complement[x]]]]]],
             q4 → member[succ[U[A[intersection[OMEGA, complement[x]]]]], x]}]] // Reverse

Out[18]= or[not[equal[x, Uchains[x]]],
          not[subclass[image[SUCC, x], x]], subclass[OMEGA, x]] == True

In[19]:= or[not[equal[x_, Uchains[x_]]],
          not[subclass[image[SUCC, x_], x_]], subclass[OMEGA, x_]] := True
```

Corollary. No set is successor-invariant and closed under unions of chains.

```
In[20]:= Map[equal[V, #] &, SubstTest[class, x, implies[member[x, u], subclass[v, x]],
          {u → intersection[invar[SUCC], fix[UCHAINS]], v → OMEGA}]]

Out[20]= equal[0, intersection[fix[UCHAINS], invar[SUCC]]] == True

In[21]:= intersection[fix[UCHAINS], invar[SUCC]] := 0
```

---

comments

Smullyan and Fitting propose using this characterization to define the class of ordinals. Doing so requires justifying the quantification over proper classes (which they discuss in detail). Here we have avoided this difficulty by proving the characterization as a theorem.

```
In[22]:=   "Raymond M. Smullyan and Melvin Fitting, Set Theory and the Continuum Problem,
            Oxford Science Publications, Clarendon Press, 1996.  See page 64.";
```

Smullyan and Fitting use the suggestive term **superinductive** for a class that is successor-invariant and closed under unions of chains. They also add the condition that the class hold the empty set. This condition is redundant because it follows from the closure under chain-unions:

```
In[23]:= implies[equal[Uchains[x], x], member[0, x]]

Out[23]= True
```