Uchains[WO]

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```
In[1]:= SetDirectory["l:"]; << goedel97.16a; << tools.m

:Package Title: goedel97.16a          2007 September 16 at 9:55 p.m.

It is now:  2007 Sep 18 at 13:30

Loading Simplification Rules

TOOLS.M                                    Revised 2007 June 25

weightlimit = 40
```

summary

The union of a chain of wellorderings need not be a wellordering.   An explicit counterexample is presented in this notebook.

derivation

Any finite total order relation is a wellordering. In particular:

```
In[2]:= SubstTest[implies, member[x, intersection[FINITE, TO]],
          member[x, WO], x → composite[id[nat[x]], inverse[S], id[nat[x]]]] // Reverse

Out[2]= WELLORDER[composite[id[nat[x]], inverse[S], id[nat[x]]]] == True

In[3]:= WELLORDER[composite[id[nat[x_]], inverse[S], id[nat[x_]]]] := True
```

Removing the nat wrapper from the above result yields this corollary:

```
In[4]:= SubstTest[implies, equal[x, nat[t]],
          WELLORDER[composite[id[x], inverse[S], id[x]]], t → x] // Reverse

Out[4]= or[not[member[x, omega]], WELLORDER[composite[id[x], inverse[S], id[x]]]] == True

In[5]:= or[not[member[x_, omega]], WELLORDER[composite[id[x_], inverse[S], id[x_]]]] := True
```

A variable-free restatement is obtained using **ReifNormality**.

```
In[6]:=  Map[empty, dif[id[omega],
            image[inverse[CART], image[inverse[IMAGE[id[inverse[S]]]], WO]]] // ReifNormality]

Out[6]=  subclass[omega,
           fix[image[inverse[CART], image[inverse[IMAGE[id[inverse[S]]]], WO]]]] == True

In[7]:=  % /. Equal → SetDelayed
```

Corollary. The class of these finite total orderings is a subclass of the class **WO**.

```
In[8]:=  SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
           {t → IMAGE[id[inverse[S]]],
            u → image[CART, id[omega]], v -> image[inverse[IMAGE[id[inverse[S]]]], WO]}] //
          Reverse

Out[8]=  subclass[image[IMAGE[id[inverse[S]]], image[CART, id[omega]]], WO] == True

In[9]:=  subclass[image[IMAGE[id[inverse[S]]], image[CART, id[omega]]], WO] := True
```

Lemma. (Chain property.)

```
In[13]:=  SubstTest[implies, member[u, chains[S]],
            member[image[IMAGE[x], u], chains[S]], {u → image[CART, id[omega]]}] // Reverse

Out[13]=  subclass[cart[image[IMAGE[x], image[CART, id[omega]]],
             image[IMAGE[x], image[CART, id[omega]]]], union[S, inverse[S]]] == True

In[14]:=  subclass[cart[image[IMAGE[x_], image[CART, id[omega]]],
             image[IMAGE[x_], image[CART, id[omega]]]], union[S, inverse[S]]] := True
```

Theorem. The union of this chain of wellorderings belongs to **Uchains[WO]**.

```
In[15]:=  SubstTest[implies, member[x, intersection[chains[S], P[t]]], member[U[x], Uchains[t]],
            {x -> image[IMAGE[id[inverse[S]]], image[CART, id[omega]]], t → WO}] // Reverse

Out[15]=  member[composite[id[omega], inverse[S], id[omega]], Uchains[WO]] == True

In[16]:=  member[composite[id[omega], inverse[S], id[omega]], Uchains[WO]] := True
```

Observation. This union is not a wellordering. (Its inverse is a wellordering.)

```
In[19]:=  WELLORDER[composite[id[omega], inverse[S], id[omega]]]

Out[19]=  False

In[20]:=  WELLORDER[composite[id[omega], S, id[omega]]]

Out[20]=  True
```

Corollary. The union of a chain of wellorderings need not be a wellordering.

```
In[17]:= Map[not, SubstTest[implies, and[member[x, y], equal[y, z]],
            member[x, z], {x -> composite[id[omega], inverse[S], id[omega]],
             y -> Uchains[WO], z → WO}]] // Reverse

Out[17]= equal[WO, Uchains[WO]] == False

In[18]:= equal[WO, Uchains[WO]] := False
```