VS and RS[x]

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2004 August 21

```
In[1]:= SetDirectory["i:"]; << goedel60.20a; << tools.m;

        :Package Title: goedel60.20a          2004 August 20 at 7:45 a.m.

        It is now:  2004 Aug 21 at 21:34

        Loading Simplification Rules

        TOOLS.M                      Revised 2004 August 11

        weightlimit = 40
```

---

summary

The vertical sections of the restrictions of a relation are among the vertical sections of the relation itself. On account of this, the function **VS** satisfies an interesting formula related to the class **RS[x]** of small restrictions. This result, and related ones are derived in this notebook. A succinct variable-free fomulation of the main formula was discovered using reification.

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some examples

Before the general theorem was discovered, these examples were derived:

```
In[2]:= composite[VS, IDP] // VSNormality

Out[2]= composite[VS, IMAGE[DUP]] == IMAGE[composite[id[SINGLETON], inverse[FIRST]]]

In[3]:= composite[VS, IMAGE[DUP]] := IMAGE[composite[id[SINGLETON], inverse[FIRST]]]

In[4]:= ImageComp[VS, IDP, V] // Reverse

Out[4]= image[VS, P[Id]] == P[SINGLETON]

In[5]:= image[VS, P[Id]] := P[SINGLETON]
```

The identity function can be replaced by an arbitrary function:

```
In[6]:= composite[VS, IMAGE[composite[id[funpart[x]], inverse[FIRST]]]] // VSNormality

Out[6]= composite[VS, IMAGE[composite[id[funpart[x]], inverse[FIRST]]]] ==
         IMAGE[composite[id[composite[SINGLETON, funpart[x]]], inverse[FIRST]]]

In[7]:= composite[VS, IMAGE[composite[id[funpart[x_]], inverse[FIRST]]]] :=
         IMAGE[composite[id[composite[SINGLETON, funpart[x]]], inverse[FIRST]]]

In[8]:= ImageComp[VS, IMAGE[composite[id[funpart[x]], inverse[FIRST]]], V] // Reverse

Out[8]= image[VS, P[funpart[x]]] == P[composite[SINGLETON, funpart[x]]]

In[9]:= image[VS, P[funpart[x_]]] := P[composite[SINGLETON, funpart[x]]]
```

The following example is the first one for which **RS[x]** appears instead of **P[x]**.

```
In[10]:= composite[VS, IMAGE[composite[id[inverse[E]], inverse[FIRST]]]] // VSNormality

Out[10]= composite[VS, IMAGE[composite[id[inverse[E]], inverse[FIRST]]]] ==
          composite[IMAGE[DUP], IMAGE[id[complement[singleton[0]]]]]

In[11]:= composite[VS, IMAGE[composite[id[inverse[E]], inverse[FIRST]]]] :=
          composite[IMAGE[DUP], IMAGE[id[complement[singleton[0]]]]]

In[12]:= ImageComp[VS, IMAGE[composite[id[inverse[E]], inverse[FIRST]]], V] // Reverse

Out[12]= image[VS, RS[inverse[E]]] == P[id[complement[singleton[0]]]]

In[13]:= image[VS, RS[inverse[E]]] := P[id[complement[singleton[0]]]]
```

---

a lemma

The following lemma shows that the restriction of **VERTSECT[x]** to the domain of **x** is unchanged when **x** is replaced by **thinpart[x]**.

```
In[14]:= composite[VERTSECT[thinpart[x]],
            id[intersection[domain[x], domain[VERTSECT[x]]]]] // RelnRenormality

Out[14]= composite[VERTSECT[thinpart[x]],
            id[intersection[domain[x], domain[VERTSECT[x]]]]] ==
          composite[VERTSECT[x], id[domain[x]]]

In[15]:= composite[VERTSECT[thinpart[x_]],
            id[intersection[domain[x_], domain[VERTSECT[x_]]]]] :=
          composite[VERTSECT[x], id[domain[x]]]
```

the general theorem

```
In[16]:= (composite[VS, IMAGE[composite[id[w], inverse[FIRST]]]] // VSNormality) /.
          w → thinpart[x]

Out[16]= composite[VS, IMAGE[composite[id[thinpart[x]], inverse[FIRST]]]] ==
          IMAGE[composite[id[composite[VERTSECT[x], id[domain[x]]]], inverse[FIRST]]]

In[17]:= composite[VS, IMAGE[composite[id[thinpart[x_]], inverse[FIRST]]]] :=
          IMAGE[composite[id[composite[VERTSECT[x], id[domain[x]]]], inverse[FIRST]]]

In[18]:= ImageComp[VS, IMAGE[composite[id[thinpart[x]], inverse[FIRST]]], V] // Reverse

Out[18]= image[VS, RS[x]] == P[composite[VERTSECT[x], id[domain[x]]]]

In[19]:= image[VS, RS[x_]] := P[composite[VERTSECT[x], id[domain[x]]]]
```

eliminating the variable x

The variable **x** in the identity deduced above can be eliminated most simply using **reify**. It is true that this result can be independently verified using **symdif** and **VSNormality**, but the latter technique requires knowing the answer in advance, whereas reification leads directly to the discovery of this formula without having any prior knowledge about it.

```
In[20]:= SubstTest[reify, x, image[v, f[x]], {v → VS, f → RS}] // Reverse

Out[20]= composite[VS, RESTRICT] == composite[inverse[S], VS]

In[21]:= composite[VS, RESTRICT] := composite[inverse[S], VS]
```

The orientation of this equation as a rewrite rule is tentative, and is based on the general notion that it is desirable to have composites with functions to be rewritten, if possible, with the function on the right. Rewrite rules for complements and intersections can be applied only when the function is on the right, unless the function is one-to-one, in which case it does not matter whether it appears on the right or left. Since relations are determined by their vertical sections, one might expect the function **VS** to be one-to-one, but technically it is not. For example:

```
In[29]:= image[inverse[VS], singleton[0]] // Normality

Out[29]= image[inverse[VS], singleton[0]] == P[complement[cart[V, V]]]
```

```
In[30]:= image[inverse[VS], singleton[0]] := P[complement[cart[V, V]]]
```

This function is nonetheless close to being one-to-one; to be precise, the restriction of **VS** to the class **P[cart[V,V]]** of all relations is one-to-one.

```
In[31]:= ONEONE[composite[VS, id[P[cart[V, V]]]]]

Out[31]= True
```