well-foundedness of strict divisibility for natural numbers

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In[1]:= **SetDirectory["i:"]; << goedel66.01a; << tools.m**

```
:Package Title: goedel66.01a          2005 February 1 at 11:00 p.m.

It is now:  2005 Feb 3 at 0:37

Loading Simplification Rules

TOOLS.M                      Revised 2005 January 7

weightlimit = 40
```

summary

It is shown in this notebook that the strict divisibility relation **intersection[Di, DIV]** is well-founded. The derivation uses a theorem discovered 2004 September 23 that can be viewed as a generalization of the theorem that lexicographic products of well founded relations are well-founded. (See the notebook **wf-u.nb**.)

a conditional rewrite rule

The following example illustrates an opportunity for a new rewrite rule.

In[2]:= **wf[composite[id[REGULAR], E]]**

Out[2]= wf[composite[id[REGULAR], E]]

The **wf** wrapper can be removed for any well-founded relation.

In[3]:= **equal[wf[x], x]**

Out[3]= WELLFOUNDED[x]

This will be added as a conditional rewrite rule.

In[4]:= **wf[x_] := x /; WELLFOUNDED[x]**

The example cited above now simplifies.

```
In[5]:= wf[composite[id[REGULAR], E]]

Out[5]= composite[id[REGULAR], E]
```

well-foundedness for strict divisibility

The main idea is to use the fact that for non-zero natural numbers, if **m** strictly divides **n**, then **m** belongs to **n**.

```
In[6]:= SubstTest[implies, subclass[u, v],
         subclass[image[w, u], image[w, v]], {u -> intersection[DIV, Di],
          v -> union[composite[id[omega], E], cart[dif[omega, set[0]], set[0]]],
          w → id[cart[V, complement[set[0]]]]}]

Out[6]= subclass[composite[id[complement[set[0]]], intersection[Di, DIV]], E] == True

In[7]:= subclass[composite[id[complement[set[0]]], intersection[Di, DIV]], E] := True
```

Since the restriction of the membership relation to natural numbers is well-founded, so is strict divisibility for non-zero numbers.

```
In[8]:= SubstTest[implies, and[subclass[u, v], WELLFOUNDED[v]], WELLFOUNDED[u],
         {u -> composite[id[complement[set[0]]], intersection[Di, DIV]],
          v → composite[id[omega], E]}]

Out[8]= WELLFOUNDED[composite[id[complement[set[0]]], intersection[Di, DIV]]] == True

In[9]:= % /. Equal → SetDelayed
```

The complement of this relation is also well-founded; no new rule is needed for this.

```
In[10]:= WELLFOUNDED[composite[id[set[0]], intersection[Di, DIV]]]

Out[10]= True
```

Lemma.

```
In[11]:= SubstTest[union, id[x], id[z], z → set[y]]

Out[11]= union[cart[set[y], set[y]], id[x]] == id[union[x, set[y]]]

In[12]:= union[cart[set[y_], set[y_]], id[x_]] := id[union[x, set[y]]]
```

Lemma.

```
In[13]:= SubstTest[composite, union[u, v], w, {u → id[set[0]],
            v -> id[complement[set[0]]], w → intersection[Di, DIV]}] // Reverse

Out[13]= union[cart[intersection[omega, complement[set[0]]], set[0]],
           composite[id[complement[set[0]]], intersection[Di, DIV]]] ==
          intersection[Di, DIV]

In[14]:= % /. Equal → SetDelayed
```

In the course of showing that the lexicographic product of well founded relations is well founded, the following generalization was discovered.

```
In[15]:= implies[subclass[composite[wf[x], wf[y]], wf[y]],
          WELLFOUNDED[union[wf[x], wf[y]]]]

Out[15]= True
```

This result can be applied here:

```
In[16]:= SubstTest[implies, subclass[composite[wf[y], wf[x]], wf[x]],
          WELLFOUNDED[union[wf[x], wf[y]]],
          {x -> cart[intersection[omega, complement[set[0]]], set[0]],
           y -> composite[id[complement[set[0]]], intersection[Di, DIV]]}]

Out[16]= WELLFOUNDED[intersection[Di, DIV]] == True

In[17]:= WELLFOUNDED[intersection[Di, DIV]] := True
```

another simplification rule

The following simplification rule was not used here, but might be useful in other applications involving explicit intersections with the diversity relation **Di**.

```
In[18]:= equal[intersection[Di, wf[x]], wf[x]]

Out[18]= True

In[19]:= intersection[Di, wf[x_]] := wf[x]
```