well-founded trichotomous relations

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2009 October 27

```
In[1]:= SetDirectory["l:"]; << goedel.09oct26a;<< tools.m

:Package Title: goedel.09oct26a              2009 October 26 at 5:45 p.m.

It is now:  2009 Oct 27 at 3:28

Loading Simplification Rules

TOOLS.M                                      Revised 2009 September 15

weightlimit = 40
```

summary

If a well-founded relation is trichotomous on the union of its domain and range, then its reflexive closure is a well-ordering.

temporary definitions

The following temporary abbreviations will be used.

```
In[2]:= irreflexive[r_, s_] := disjoint[s, fix[r]]

In[3]:= transitive[r_, s_] := TRANSITIVE[restrict[r, s, s]]

In[4]:= trichotomous[r_, s_] := subclass[cart[s, s], union[Id, r, inverse[r]]]

In[5]:= strictwellorder[r_, s_] := and[trichotomous[r, s], WELLFOUNDED[composite[id[s], r]]]
```

Reference:

```
In[6]:= "Patrick Suppes, Axiomatic Set Theory, Dover Publications, New York, 1972.";
```

derivation

Lemma. (Corollary of Suppes's Theorem 62.)

```
In[13]:= SubstTest[implies, strictwellorder[r, s],
           transitive[r, s], {r → wf[x], s → udora[wf[x]]}] // Reverse

Out[13]= or[not[subclass[
              cart[union[domain[wf[x]], range[wf[x]]], union[domain[wf[x]], range[wf[x]]]],
              union[Id, inverse[wf[x]], wf[x]]]], TRANSITIVE[wf[x]]] == True

In[14]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. If a strict order is trichotomous on the union of its domain and range, then its reflexive closure is a total-order.

```
In[21]:= SubstTest[implies, and[equal[0, fix[r]],
            subclass[cart[union[domain[r], range[r]], union[domain[r], range[r]]],
             union[Id, r, inverse[r]]], TRANSITIVE[r]],
           TOTALORDER[union[r, id[union[domain[r], range[r]]]]], r → wf[x]] // Reverse

Out[21]= or[not[subclass[
              cart[union[domain[wf[x]], range[wf[x]]], union[domain[wf[x]], range[wf[x]]]],
              union[Id, inverse[wf[x]], wf[x]]]], not[TRANSITIVE[wf[x]]],
           TOTALORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]] == True

In[22]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. A total order whose strict part is well-founded is a well-ordering.

```
In[26]:= Map[implies[TOTALORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]], #] &,
          SubstTest[and, TOTALORDER[t], WELLFOUNDED[intersection[Di, t]],
           t -> union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]]

Out[26]= or[not[TOTALORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]],
           WELLORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]] == True

In[27]:= (% /. x → x_) /. Equal → SetDelayed
```

Lemma. (Combining the preceding lemmas.)

```
In[30]:= Map[not, SubstTest[and, implies[p1, p2],
            implies[and[p1, p2], p3], implies[p3, p4], not[implies[p1, p4]],
            {p1 → trichotomous[wf[x], udora[wf[x]]], p2 -> TRANSITIVE[wf[x]],
             p3 -> TOTALORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]],
             p4 → WELLORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]}]] // Reverse

Out[30]= or[not[subclass[cart[union[domain[wf[x]], range[wf[x]]],
               union[domain[wf[x]], range[wf[x]]]], union[Id, inverse[wf[x]], wf[x]]]],
           WELLORDER[union[id[union[domain[wf[x]], range[wf[x]]]], wf[x]]]] == True

In[31]:= (% /. x → x_) /. Equal → SetDelayed
```

The main theorem is obtained by removing the **wf** wrapper.

Theorem. If a well-founded relation is trichotomous on the union of its domain and range, then its reflexive closure is a well-ordering.

```
In[32]:= SubstTest[implies, and[equal[x, wf[t]], trichotomous[x, udora[x]]],
           WELLORDER[union[x, id[udora[x]]]], t → x] // Reverse

Out[32]= or[not[subclass[cart[union[domain[x], range[x]], union[domain[x], range[x]]],
             union[Id, x, inverse[x]]]], not[WELLFOUNDED[x]],
           WELLORDER[union[x, id[union[domain[x], range[x]]]]]] == True

In[33]:= or[not[subclass[cart[union[domain[x_], range[x_]], union[domain[x_], range[x_]]],
             union[Id, x_, inverse[x_]]]], not[WELLFOUNDED[x_]],
           WELLORDER[union[x_, id[union[domain[x_], range[x_]]]]]] := True
```