counterexample related to Zermelo's theorem

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```
In[1]:= SetDirectory["l:"]; << goedel77.07a; << tools.m

:Package Title: goedel77.07a                    2006 January 7 at 1:40 p.m.

It is now:  2006 Jan 8 at 5:24

Loading Simplification Rules

TOOLS.M                        Revised 2006 January 2

weightlimit = 40
```

summary

Zermelo's fixed point theorem says:

```
In[2]:= implies[and[member[x, map[P[y], P[y]]], subcommute[x, S]], not[empty[fix[x]]]]

Out[2]= True
```

In this notebook it is shown that this theorem does not generalize to proper classes.

a restatement of Zermelo's theorem

Lemma.

```
In[3]:= SubstTest[subclass, intersection[u, v], u,
          {u -> image[inverse[IMAGE[FIRST]], set[x]], v → intersection[FUNS, P[cart[x, y]]]}]

Out[3]= subclass[image[IMAGE[FIRST], map[x, y]], set[x]] == True

In[4]:= subclass[image[IMAGE[FIRST], map[x_, y_]], set[x_]] := True
```

Theorem.

```
In[5]:= equal[map[x, x],
          intersection[image[inverse[IMAGE[FIRST]], set[x]], U[image[MAP, Id]]]] // AssertTest

Out[5]= equal[intersection[image[inverse[IMAGE[FIRST]], set[x]], U[image[MAP, Id]]],
          map[x, x]] == True

In[6]:= intersection[image[inverse[IMAGE[FIRST]], set[x_]], U[image[MAP, Id]]] := map[x, x]
```

Restatement of Zermelo's theorem.

```
In[7]:= Map[implies[and[#, subcommute[x, S]], not[empty[fix[x]]]] &,
          SubstTest[member, x, intersection[u, v],
           {u -> image[inverse[IMAGE[FIRST]], set[P[y]]], v -> U[image[MAP, Id]]}]] // Reverse

Out[7]= or[not[equal[0, fix[x]]], not[equal[domain[x], P[y]]], not[FUNCTION[x]],
          not[member[x, V]], not[subclass[composite[x, S], composite[S, x]]],
          not[subclass[range[x], domain[x]]]] == True

In[8]:= or[not[equal[0, fix[x_]]], not[equal[domain[x_], P[y_]]], not[FUNCTION[x_]],
          not[member[x_, V]], not[subclass[composite[x_, S], composite[S, x_]]],
          not[subclass[range[x_], domain[x_]]]] := True
```

In the next section it is shown that one cannot omit the sethood literal.

the counterexample

The counterexample is:

```
In[10]:= or[not[equal[0, fix[x]]], not[equal[domain[x], P[y]]],
           not[FUNCTION[x]], not[subclass[composite[x, S], composite[S, x]]],
           not[subclass[range[x], domain[x]]]] /. {x → POWER, y → V}

Out[10]= False
```

This does not contradict Zermelo's theorem because the function **POWER** is not a set. It does not belong to any class.

```
In[14]:= member[POWER, x]

Out[14]= False
```

Except for the sethood hypothesis, all the other hypotheses of Zermelo's theorem are satisfied:

```
In[12]:= and[equal[domain[x], P[y]], FUNCTION[x],
           subcommute[x, S], subclass[range[x], domain[x]]] /. {x → POWER, y → V}

Out[12]= True
```

The conclusion is false. Cantor's theorem implies that **POWER** has no fixed point.

```
In[13]:= fix[POWER]

Out[13]= 0
```