Zermelo theorem restated

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```
In[1]:= SetDirectory["l:"]; << goedel77.06a; << tools.m

        :Package Title: goedel77.06a      2006 January 6 at 8:45 a.m.

        It is now:  2006 Jan 7 at 13:4

        Loading Simplification Rules

        TOOLS.M                       Revised 2006 January 2

        weightlimit = 40
```

summary

A special case of complete lattice fixed point theorem is the Zermelo fixed point theorem. A variant of this theorem is derived in this notebook, starting from the following version, which is already available:

```
In[4]:= implies[and[member[x, map[P[y], P[y]]], subcommute[x, S]], not[empty[fix[x]]]]

Out[4]= True
```

The version to be derived here is similar, but replaces **subcommute** with **monotone**.

```
In[3]:= monotone[x, S, S]

Out[3]= subclass[composite[x, S, inverse[x]], S]
```

A variable-free version of this theorem is also derived.

derivation

Theorem.

```
In[17]:= Map[not, SubstTest[and, implies[p1, p3], implies[p1, p4], implies[p4, p5],
           implies[and[p2, p3, p5], p6], implies[and[p1, p6], p7], not[implies[and[p1, p2], p7]],
           {p1 -> member[x, map[P[y], P[y]]], p2 -> monotone[x, S, S], p3 → FUNCTION[x],
            p4 → equal[domain[x], P[y]], p5 → equal[image[inverse[S], domain[x]], domain[x]],
            p6 → subcommute[x, S], p7 → not[empty[fix[x]]]}]]

Out[17]= or[not[equal[0, fix[x]]], not[member[x, map[P[y], P[y]]]],
           not[subclass[composite[x, S, inverse[x]], S]]] == True
```

```
In[18]:= or[not[equal[0, fix[x_]]], not[member[x_, map[P[y_], P[y_]]]],
           not[subclass[composite[x_, S, inverse[x_]], S]]] := True
```

variable-free

Lemma.

```
In[20]:= Map[equal[V, #] &, SubstTest[class, y, or[not[equal[0, fix[y]]], not[member[y, z]],
             not[subclass[composite[y, S, inverse[y]], S]]], z → map[P[x], P[x]]]] // Reverse

Out[20]= equal[0, intersection[cliques[complement[cross[S, complement[S]]]],
            map[P[x], P[x]], P[union[Di, complement[cart[V, V]]]]]] == True
```

```
In[22]:= intersection[cliques[complement[cross[S, complement[S]]]],
           map[P[x_], P[x_]], P[union[Di, complement[cart[V, V]]]]] := 0
```

Simplification:

```
In[24]:= AssInt[intersection[cliques[complement[cross[S, complement[S]]]], map[P[x], P[x]],
          P[union[Di, complement[cart[V, V]]]]], P[Di]]

Out[24]= intersection[cliques[complement[cross[S, complement[S]]]], map[P[x], P[x]], P[Di]] == 0
```

```
In[25]:= intersection[cliques[complement[cross[S, complement[S]]]],
           map[P[x_], P[x_]], P[Di]] := 0
```

The variable **x** can also be eliminated, yielding this simple restatement of Zermelo's theorem.

```
In[31]:= Map[U[range[VERTSECT[#]]] &,
           SubstTest[reify, x, intersection[cliques[complement[cross[S, complement[S]]]],
             map[P[x], P[x]], y], y -> P[Di]]] // Reverse

Out[31]= intersection[cliques[complement[cross[S, complement[S]]]],
           P[Di], U[image[MAP, id[range[POWER]]]]] == 0
```

```
In[32]:= intersection[cliques[complement[cross[S, complement[S]]]],
           P[Di], U[image[MAP, id[range[POWER]]]]] := 0
```

This rewrite rule says that the class of monotone fixed-point free relations is disjoint from the class of mappings of a power set to itself.