Kuratowski's construction of ordered pairs

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```
In[1]:= SetDirectory["i:"]; << goedel65.17a; << tools.m

:Package Title: goedel65.17a          2005 January 17 at 7:40 p.m.

It is now:  2005 Jan 18 at 4:22

Loading Simplification Rules

TOOLS.M                     Revised 2005 January 7

weightlimit = 40
```

summary

The **GOEDEL** program does not assume Kuratowski's construction for ordered pairs, but this construction is nonetheless useful for deriving properties of cartesian products. In this notebook, the sethood rule for cartesian products is removed, and then rederived using the function **KURA** which maps ordered pairs to Kuratowski's model for them:

```
In[2]:= lambda[pair[x, y], set[set[x], set[x, y]]]

Out[2]= KURA
```

comment on notation

The class **set[x, y, ...]** is the class of all sets **w** such that **w = x** or **w = y** or **...** . The older notations **singleton[x]** and **pairset[x, y]** are still available for the case of one or two arguments:

```
In[3]:= singleton[x]

Out[3]= set[x]

In[4]:= pairset[x, y]

Out[4]= set[x, y]
```

The function **SINGLETON** maps sets to their singletons and the function **PAIRSET** maps ordered pairs to pairsets.

```
In[5]:= lambda[x, set[x]]

Out[5]= SINGLETON

In[6]:= lambda[pair[x, y], set[x, y]]

Out[6]= PAIRSET
```

---

some properties of the function KURA

The function **KURA** is a one-to-one function with domain **cart[V,V]**.

```
In[7]:= ONEONE[KURA]

Out[7]= True

In[8]:= domain[KURA]

Out[8]= cart[V, V]
```

The function **KURA** can be written in terms of the functions **PAIRSET** and **SINGLETON** as follows:

```
In[9]:= composite[PAIRSET, cross[composite[SINGLETON, FIRST], PAIRSET], DUP]

Out[9]= KURA
```

This formula can be used to derive the following needed formula for the image of a cartesian product under **KURA**.

```
In[10]:= ImageComp[PAIRSET,
          composite[cross[composite[SINGLETON, FIRST], PAIRSET], DUP], cart[x, y]]

Out[10]= image[KURA, cart[x, y]] == image[PAIRSET,
           composite[PAIRSET, id[cart[x, y]], inverse[FIRST], inverse[SINGLETON]]]

In[11]:= image[KURA, cart[x_, y_]] := image[PAIRSET,
           composite[PAIRSET, id[cart[x, y]], inverse[FIRST], inverse[SINGLETON]]]
```

The following corollary is basic:

```
In[12]:= U[U[image[KURA, cart[x, y]]]]

Out[12]= union[intersection[x, image[V, y]], intersection[y, image[V, x]]]
```

sethood of cartesian products

The **GOEDEL** program contains this sethood rule for cartesian products:

```
In[13]:= member[cart[x, y], V]

Out[13]= or[and[member[x, V], member[y, V]], equal[0, x], equal[0, y]]
```

This rule is now removed, and will be quickly rederived using the function **KURA**.

```
In[14]:= member[cart[x_, y_], V] =.
```

The basic tool needed is the axiom of replacement, from which this corollary can be derived, using the wrapper **oopart** for a generic one-to-one function.

```
In[15]:= member[image[oopart[x], y], V]

Out[15]= member[intersection[y, domain[oopart[x]]], V]
```

The rederivation of the sethood rule for cartesian products is accomplished in a single step:

```
In[16]:= SubstTest[member, image[oopart[z], w], V, {w → cart[x, y], z → KURA}] // Reverse

Out[16]= member[cart[x, y], V] ==
          or[and[member[x, V], member[y, V]], equal[0, x], equal[0, y]]
```