

Putnam exam question

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2001 December 5

■ The question.

The Putnam exam contained the following question. Suppose $\mathbf{b}[\mathbf{x},\mathbf{y}]$ is a binary operation on a set, such that $\mathbf{b}[\mathbf{x},\mathbf{b}[\mathbf{y},\mathbf{x}]] = \mathbf{y}$ for all elements \mathbf{x}, \mathbf{y} of the set. Show that this operation also satisfies $\mathbf{b}[\mathbf{b}[\mathbf{x},\mathbf{y}],\mathbf{x}] = \mathbf{x}$ for all elements of the set.

■ The *Mathematica* solution.

Begin by formulating the hypothesis as a rewrite rule.

```
b[x_, b[y_, x_]] := y
```

The following **SubstTest** allows one to easily compare the result of different orders of evaluation.

```
SubstTest[f_, x_, r_] := Equal[f@@({x}/.r), (f@@{x}/.r)]
```

To use the test, let **f** be the head of an expression to which the hypothesis applies, take **x** to be a list of expressions which are the arguments that occur in this expression, and for **r** pick an appropriate substitution to make. This is not automated reasoning; some care must be exercised in choosing this substitution.

```
SubstTest[b, z, b[y, z], z -> b[x, y]]
```

```
b[b[x, y], x] == y
```

A **Trace** will reveal what actually happened, but this is usually too detailed to provide much insight. It is slightly better for this purpose to use the command **On**.

```
On[b]
```

```
SubstTest[b, z, b[y, z], z -> b[x, y]]
```

```
b::trace : b[y, b[x, y]] -> x.
```

```
b::trace : b[z, b[y, z]] -> y.
```

```
b[b[x, y], x] == y
```

```
Off[]
```

■ A few more details.

What **SubstTest** does is to compare two different orders of evaluation: one order is the "lazy" evaluation

```
b[z, b[y, z]] /. z -> b[x, y]
```

```
y
```

The other order is the "eager" order, in which the arguments are evaluated first.

```
Apply[b, (List[z, b[y, z]] /. z -> b[x, y])]
```

```
b[b[x, y], x]
```

The default for *Mathematica* is to use this latter order. If we make the substitution by hand, and ask *Mathematica* to evaluate, this is what we find:

```
b[b[x, y], b[y, b[x, y]]]
```

```
b[b[x, y], x]
```