numbers unconstrued

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```
In[1]:= SetDirectory["l:"]; << goedel.10may18a;<< tools.m

        :Package Title: goedel.10may18a              2010 May 18 at 5:55 a.m.

        It is now:  2010 May 19 at 8:6

        Loading Simplification Rules

        TOOLS.M                                      Revised 2010 February 26

        weightlimit = 40
```

summary

The existence of a set **omega =** ω of natural numbers depends on the axiom of infinity, but the existence of individual natural numbers **0, {0}, {0,{0}}, ...** does not. The usual definition of the class **omega =** ω of natural numbers is:

```
In[2]:= A[class[z, and[member[0, z], invariant[SUCC, z]]]]

Out[2]= omega
```

If there were no infinite sets, this definition would reduce to ω **= A[0] = V**. It follows from this that he usual definition of natural numbers as members of the class **omega =** ω of natural numbers only works if one assumes the axiom of infinity. This notebook concerns Quine's definition of the class of natural numbers, which does not require the axiom of infinity. See Chapter 4 starting on page 74 in the following reference.

```
"Willard van Orman Quine, Set Theory and its Logic, The Belknap Press of Harvard
 University Press, Cambridge, Mass., 1963.  Third edition (paperback), 1971.";
```

The **GOEDEL** program itself is not particularly suitable for developing mathematics in the absence of the axiom of infinity, but one could use McCune's program **Otter** to do so. All that will be done here is to derive the fact that Quine's definition agrees with the usual one when one does assume the axiom of infinity.

derivation

Quine's basic idea is to base the definition of natural numbers on invariance under **inverse[SUCC]** instead of invariance under the successor function **SUCC**.

Lemma. A temporary simplification rule.

```
In[15]:= image[inverse[SINGLETON],
           image[inverse[HULL[invar[inverse[SUCC]]]], P[complement[set[0]]]]] // Normality

Out[15]= image[inverse[SINGLETON],
           image[inverse[HULL[invar[inverse[SUCC]]]], P[complement[set[0]]]]] ==
          core[invar[inverse[SUCC]], complement[set[0]]]

In[16]:= % /. Equal → SetDelayed
```

The equivalence of Quine's definition with the usual one in the presence of the axiom of infinity depends on the following:

Theorem.

```
In[17]:= Map[complement, IminComp[inverse[E],
            composite[HULL[invar[inverse[SUCC]]], SINGLETON], set[0]]] // Reverse

Out[17]= core[invar[inverse[SUCC]], complement[set[0]]] == complement[omega]

In[18]:= core[invar[inverse[SUCC]], complement[set[0]]] := complement[omega]
```

Quine's definition of the class of natural numbers is this:

```
In[19]:= class[x,
          forall[z, implies[and[member[x, z], invariant[inverse[SUCC], z]], member[0, z]]]]

Out[19]= omega
```

a quasi-order

Quine defines the following relation which is reflexive and transitive.

```
In[20]:= class[pair[y, x],
          forall[z, implies[and[member[x, z], invariant[inverse[SUCC], z]], member[y, z]]]]

Out[20]= union[Id, trv[SUCC]]

In[21]:= REFLEXIVE[union[Id, trv[SUCC]]]

Out[21]= True

In[22]:= TRANSITIVE[union[Id, trv[SUCC]]]

Out[22]= True
```