transforms of well-founded relations

Johan G. F. Belinfante
2004 September 23

```
In[1]:= SetDirectory["i:"]; << goedel61.21a; << tools.m

        :Package Title: goedel61.21a          2004 September 21 at 4:45 p.m.

        It is now:  2004 Sep 23 at 13:12

        Loading Simplification Rules

        TOOLS.M                      Revised 2004 September 18

        weightlimit = 40
```

summary

If **x** is a function and **y** is a well-founded relation, then **composite[inverse[x],y],x]** is well-founded. This result can be used to obtain a simple proof of the theorem that **cross[x,y]** is well-founded when **x** or **y** is well-founded.

derivation

The derivation is especially succinct when wrappers are used to represent the hypotheses.

```
In[2]:= SubstTest[equal, singleton[0], subvar[w],
          w -> composite[inverse[funpart[x]], wf[y], funpart[x]]] // Reverse

Out[2]= WELLFOUNDED[composite[inverse[funpart[x]], wf[y], funpart[x]]] == True

In[3]:= WELLFOUNDED[composite[inverse[funpart[x_]], wf[y_], funpart[x_]]] := True
```

The wrappers are easily replaced with literals for the hypotheses:

```
In[4]:= SubstTest[implies, and[equal[u, funpart[x]], equal[v, wf[y]]],
         WELLFOUNDED[composite[inverse[u], v, u]], {u → x, v → y}]

Out[4]= or[not[FUNCTION[x]], not[WELLFOUNDED[y]],
          WELLFOUNDED[composite[inverse[x], y, x]]] == True
```

```
In[5]:= or[not[FUNCTION[x_]], not[WELLFOUNDED[y_]],
          WELLFOUNDED[composite[inverse[x_], y_, x_]]] := True
```

corollary

```
In[6]:= SubstTest[WELLFOUNDED,
         composite[inverse[funpart[w]], wf[x], funpart[w]], w → FIRST]

Out[6]= WELLFOUNDED[composite[inverse[FIRST], wf[x], FIRST]] == True

In[7]:= WELLFOUNDED[composite[inverse[FIRST], wf[x_], FIRST]] := True
```

It follows that **cross[wf[x],V]** is well-founded:

```
In[8]:= cross[wf[x], V]

Out[8]= composite[inverse[FIRST], wf[x], FIRST]
```

Since any subclass of a well-founded relation is well-founded, it now follows that **cross[wf[x],y]** is well-founded for all **y**.

```
In[9]:= subclass[cross[wf[x], y], cross[wf[x], V]]

Out[9]= True
```

Similarly, one can show that **cross[x, wf[y]]** is well-founded using the following fact:

```
In[10]:= SubstTest[WELLFOUNDED,
          composite[inverse[funpart[w]], wf[x], funpart[w]], w → SECOND]

Out[10]= WELLFOUNDED[composite[inverse[SECOND], wf[x], SECOND]] == True

In[11]:= WELLFOUNDED[composite[inverse[SECOND], wf[x_], SECOND]] := True
```

One can use the following fact to show that **cross[x,y]** is well-founded if and only if **cross[y,x]** is well-founded.

```
In[12]:= SubstTest[WELLFOUNDED,
          composite[inverse[funpart[w]], wf[x], funpart[w]], w → SWAP]

Out[12]= WELLFOUNDED[composite[SWAP, wf[x], SWAP]] == True

In[13]:= WELLFOUNDED[composite[SWAP, wf[x_], SWAP]] := True
```