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Grand Canonical Evolution for the Kac Model

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Abstract We study a model of random colliding particles interacting with an infinite reservoir at fixed temperature and chemical potential. Interaction between the particles is modeled via a Kac master equation [9]. Moreover, particles can leave the system toward the reservoir or enter the system from the reservoir. The system admits a unique steady state given by the Grand Canonical Ensemble at temperature $T = \beta^{-1}$ and chemical potential χ. We show that any initial state converges exponentially to equilibrium by computing the spectral gap of the generator in a suitable L^2 space and by showing exponential decrease of the relative entropy with respect to the steady state. We also show propagation of chaos and thus the validity of a Boltzmann-Kac type equation for the particle density in the infinite system limit.

Keywords Kac model, Approach to equilibrium, Particle reservoir

1 Introduction

In 1955, Mark Kac [9] introduced a simple model to study the evolution of a gas of N particles undergoing pairwise collisions. Instead of following the deterministic evolution of the particles until a collision takes place, he considered particles that collide at random times with every particle undergoing, on average, a given number of collisions per unit time. Moreover, when a collision takes place, the energy of the two particles is randomly redistributed between them. In such a situation, one can neglect the position of the particles and focus on their velocities. Finally, to obtain a model

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as simple as possible, he considered particles that move in one spatial dimension. This naturally leads to an evolution governed by a master equation for the probability distribution $f(\underline{v}_N)$, where $\underline{v}_N \in \mathbb{R}^N$ describes the velocities of the particles.

The study of the Kac Master equation has been very useful to clarify and investigate notions and conjectures arising from the kinetic theory of diluted gases. We refer the reader to Kac's original works [9] and [10] for extensive discussion.

Kac's master equation also provides a natural setting to study approach to equilibrium. In the case of the standard Kac model [9], equilibrium is represented by the uniform distribution on the surface of given kinetic energy. Uniform convergence in the sense of the L^2 gap was conjectured by Kac and it was established in [8] while the gap was explicitly computed in [3].

A more natural way to define approach to equilibrium is via the entropy. This provides a better setting since the entropy, in general, grows only linearly with the number of particles. There is no result of exponential decay of entropy with a rate that is uniform in N for the original Kac model. Moreover, estimates of the entropy production rate seem to point to a slow decay, at least for short times, see [12,6].

In [2], the authors studied a Kac model with N particles brought to equilibrium via a Maxwellian thermostat, i.e. an infinite heat reservoir at fixed temperature $T = \beta^{-1}$. Particles in the system evolve according to the standard Kac collision process. Moreover they collide with particles in the thermostat and exchange energy, but there is no exchange of particles between the system and the reservoir. They proved that the system admits as a unique steady state the Canonical Ensemble, i.e. in the steady state the probability distribution $f(\underline{v}_N)$ is the Maxwellian distribution at temperature T. Moreover, the steady state is approached exponentially fast and uniformly in N, both in the sense of the spectral gap, in a suitable L^2 space, and in the sense of the relative entropy. They also adapted the McKean proof [11] of propagation of chaos and obtained a Boltzmann-Kac type effective equation for the evolution of the one particle marginal in the limit for $N \to \infty$.

In the present work, we study a different way to bring the system to equilibrium. Like in [2], the system evolves through random collision and interacts with an infinite reservoir at given temperature T; however, the system and the reservoir are allowed to exchange particles. More precisely, at random times a particle in the system can leave while, still at random times, a particle can enter the system from the reservoir. Since the reservoir is infinite, no particle can enter or leave the system more than once. We show that this new evolution admits as its unique steady state the Grand Canonical Ensemble. This means that the probability that the system contains N particles is given by a Poisson distribution while the probability distribution on the velocities, given the number of particles, is the Maxwellian at temperature T.

We also study the approach to equilibrium in a suitable L^2 space and in relative entropy. In both cases, we show that the rate of approach is uniform in the average number of particles. Finally, we look at the emergence of an effective evolution for the particle density in the limit of a large system. This requires some adaptation of the concept of propagation of chaos since the number of particles in the system is not constant. Adapting the proof in [11], we show that the relative particle density, defined in (9) and (10) below, satisfies a Boltzmann-Kac type of equation.

The rest of the paper is organized as follows. In section 2, we present the model and state our main results. Section 3 contains all the proofs, while in section 4 we report some open problems and present possible areas of future work.

2 Model and Results

We consider a system of particles in one space dimension interacting with an infinite reservoir with which it exchanges particles. Since the number of particles in the system is not constant, the phase space is given by $\mathscr{R} = \bigcup_{N=0}^{\infty} \mathbb{R}^N$.

The evolution of the system is governed by three separate random processes. First, at exponentially distributed times a particle is added to the system with a velocity randomly chosen from a Maxwellian distribution at temperature T. To simplify notation we chose $T^{-1} = 2\pi$. Second, also at exponentially distributed times, a particle is chosen at random to exit the system and disappear forever with no chance of reentry. Finally, particles in the system are selected at random to undergo a standard Kac collision.

Let $\mathbf{f} = (f_N)_{N=0}^{\infty}$ be a probability distribution in $L^1_s(\mathscr{R}) = \bigoplus_{N=0}^{\infty} L^1_s(\mathbb{R}^N)$ where $f_N \in L^1_s(\mathbb{R}^N)$ is symmetric under permutation of the v_i. Here $f_N(\underline{v}_N)$ represents the probability of finding N particles in the system with velocities $\underline{v}_N = (v_1, \dots, v_N)$.

The master equation for the evolution is thus given by

$$\frac{d}{dt}\mathbf{f} = \mathscr{L}[\mathbf{f}] := \mu(\mathscr{I}[\mathbf{f}] - \mathbf{f}) + \rho(\mathscr{O}[\mathbf{f}] - \mathscr{N}[\mathbf{f}]) + \tilde{\lambda}\mathscr{K}[\mathbf{f}] \tag{1}$$

where $\mathscr{I}$ is the *in* operator that represents the effect of introducing a particle into the system and, after symmetrization, is given by

$$(\mathscr{I}\mathbf{f})_N(v_1, \dots, v_N) = \frac{1}{N}\sum_{i=1}^{N} e^{-\pi v_i^2} f_{N-1}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$$

while $\mathscr{O}$ is the *out* operator that represents the effect of a random particle leaving the system

$$(\mathscr{O}\mathbf{f})_N(v_1, \dots, v_N) = \sum_{i=1}^{N+1} \int dw f_{N+1}(v_1, \dots, v_{i-1}, w, v_i, \dots, v_N)$$

and

$$(\mathscr{N}\mathbf{f})_N(v_1, \dots, v_N) = N f_N(v_1, \dots, v_N)\,.$$

These also define the *thermostat* operator

$$\mathscr{T} := \mu(\mathscr{I} - \mathrm{Id}) + \rho(\mathscr{O} - \mathscr{N})\,. \tag{2}$$

These definitions imply that, in every time interval dt, there is a probability μdt of a particle being added to the system. This probability is independent of the number of particles already in the system. In the same time interval, every particle in the system has a probability ρdt of leaving the system, which is, again, independent of the number of particles in the system.

Finally $\mathscr{K}$ represents the effect of the particles collisions and it is given by

$$(\mathscr{K}\mathbf{f})_N = K_N f_N := \sum_{1\le i<j\le N} (R_{i,j} - I) f_N$$

where $R_{i,j}$ represents the effect of a collision between particles i and j

$$R_{i,j} f_N = \int \frac{d\theta}{2\pi} f_N(v_1,\dots,v_i\cos\theta + v_j\sin\theta,\dots,v_i\sin\theta - v_j\cos\theta,\dots,v_N)\,.$$

In this way, the probability that two given particles, i and j, suffer a collision in an interval dt is proportional to $\tilde{\lambda}$ and does not depend on the number of particle in the system.

If we assume that the average number of particles in the system is finite, that is

$$\langle \mathscr{N}\mathbf{f}\rangle := \sum_{N=0}^{\infty} \int N f_N(\underline{v}_N) d\underline{v}_N = \sum_{N=0}^{\infty} N \bar{f}_N < \infty$$

we get

$$\lim_{N\to\infty} \frac{d}{dt}\sum_{i=0}^{N} \bar{f}_i = \lim_{N\to\infty} \sum_{i=0}^{N} \left(\mu \bar{f}_{i-1} + (i+1)\rho \bar{f}_{i+1} - (\mu + i\rho)\bar{f}_i\right) = \lim_{N\to\infty} ((N+1)\rho \bar{f}_{N+1} - \mu \bar{f}_N) = 0$$

so that, the evolution given by (1) preserves probability distributions.

It is not hard to see that the unique steady state $\boldsymbol{\Gamma}$ of the evolution is given by

$$(\boldsymbol{\Gamma})_N(\underline{v}_N) = \left(\frac{\mu}{\rho}\right)^N \frac{e^{-\frac{\mu}{\rho}}}{N!} e^{-\pi|\underline{v}_N|^2} := a_N \gamma_N(\underline{v}_N)$$

where $\gamma_N(\underline{v}_N) = \prod_{i=1}^N \gamma(v_i)$, with $\gamma(v) = e^{-\pi v^2}$, is the Maxwellian distribution in N dimension while $a_N = \left(\frac{\mu}{\rho}\right)^N \frac{e^{-\frac{\mu}{\rho}}}{N!}$ is a Poisson distribution on $\mathbb{N}$. We observe that $\boldsymbol{\Gamma}$ is a Grand Canonical Ensemble with temperature $T = \beta^{-1} = 1/2\pi$, chemical potential $\chi = (2\pi)^{-1}\log(\rho/\mu)$, and average number of particles $\langle \mathscr{N}\boldsymbol{\Gamma}\rangle = \mu/\rho$.

As discussed in the introduction, we are interested in properties that are uniform in the average number of particles $\langle \mathscr{N}\boldsymbol{\Gamma}\rangle$. For this to make sense, we must also assume that the collision rate between particles decreases as the average number of particles increases. It is thus natural to take:

$$\tilde{\lambda} = \lambda \frac{\rho}{\mu}\,.$$

One way to study the approach of an initial state $\mathbf{f}$ toward $\boldsymbol{\Gamma}$ is by computing the spectral gap of $\mathscr{L}$. Since $\mathscr{L}$ is not self adjoint on $L^2_s(\mathscr{R})$ we perform a ground state transformation setting

$$f_N := a_N \gamma_N h_N\,. \tag{3}$$

We will express (3) as $\mathbf{f} = \boldsymbol{\Gamma}\mathbf{h}$. Inserting the above definition in (1) we get

$$\frac{d}{dt}\mathbf{h} = \widetilde{\mathscr{L}}\mathbf{h} := \rho(\mathscr{P}^+\mathbf{h} - \mathscr{N}\mathbf{h}) + \mu(\mathscr{P}^-\mathbf{h} - \mathbf{h}) + \tilde{\lambda}\mathscr{K}\mathbf{h} \tag{4}$$

where we have set

$$(\mathscr{P}^+\mathbf{h})_N = \sum_{i=1}^{N} h_{N-1}(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_N)$$
$$(\mathscr{P}^-\mathbf{h})_N = \frac{1}{N+1}\sum_{i=1}^{N+1}\int dw e^{-\pi w^2} h_{N+1}(v_1,\ldots,v_{i-1},w,v_i,\ldots,v_N)$$

The following Theorem shows that $\widetilde{\mathscr{L}}$ defines an evolution on the Hilbert space

$$L^2_s(\mathscr{R},\boldsymbol{\Gamma}) = \bigoplus_{N=0}^{\infty} L^2_s(\mathbb{R}^N, a_N\gamma_N(\underline{v}_N))$$

of state $\mathbf{h} = (h_1,h_2,\ldots)$ with $h_N(\underline{v}_N)$ symmetric under permutation of the v_i and defined by the scalar product

$$(\mathbf{h}_1,\mathbf{h}_2) = \sum_{N=0}^{\infty} a_N \int_{\mathbb{R}^N} h_{1,N}(\underline{v}_N)h_{2,N}(\underline{v}_N)\gamma_N(\underline{v}_N)d\underline{v}_N\,.$$

Theorem 1 *The generator $\widetilde{\mathscr{L}}$ is self adjoint and non-positive definite on $L^2_s(\mathscr{R},\boldsymbol{\Gamma})$. Furthermore, if we define*

$$\Delta = \sup\{(\mathbf{h},\widetilde{\mathscr{L}}\mathbf{h})\,|\,\|\mathbf{h}\|_2 = 1, \mathbf{h}\perp\mathbf{E}_0\}$$

where $\|\mathbf{h}\|_2 = (\mathbf{h},\mathbf{h})$ *and* $\mathbf{E}_0 = \mathrm{span}(\mathbf{e}^0)$ *with* $(\mathbf{e}^0)_N \equiv 1$*, we get*

$$\Delta = -\rho\,.$$

The eigenspace associated with Δ is $\mathbf{E}_1 = \mathrm{span}(\mathbf{e}_1,\mathbf{e}_{(0,0,1)})$ *with* $\mathbf{e}_1 = \sqrt{\frac{\rho}{\mu}}\mathscr{P}^+\mathbf{e}^0 - \sqrt{\frac{\mu}{\rho}}\mathbf{e}^0$ *while*

$$(\mathbf{e}_{(0,0,1)})_N(\underline{v}_N) = \sqrt{\frac{\rho}{\mu}}\sum_{i=1}^{N}(v_i^2 - 2\pi)\,.$$

Proof. See section 3.1.

Due to the invariance of degree 2 even polynomials under the Kac collision operator $\mathscr{K}$, Theorem 1 shows that the spectral gap of the generator $\widetilde{\mathscr{L}}$ is completely determined by the presence of the reservoir. Moreover, observe that the gap depends only on ρ and is independent of μ.

Like in [2], to see the effect of the Kac collision operator $\mathscr{K}$, we have to look at the second gap, defined as

$$\Delta_2 = \sup\{(\mathbf{h},\widetilde{\mathscr{L}}\mathbf{h})\,|\,\|\mathbf{h}\| = 1, \mathbf{h}\perp\mathbf{E}_0\oplus\mathbf{E}_1\}\,.$$

Theorem 2 *If*

$$\rho > \frac{\lambda}{4} + 2\lambda\sqrt{\frac{\rho}{\mu}} \quad \text{and} \quad \frac{\mu}{\rho} > K \tag{5}$$

then we have

$$-\rho - \frac{\lambda}{4} \le \Delta_2 < -\rho - \frac{\lambda}{4} + C\lambda\sqrt{\frac{\rho}{\mu}} \tag{6}$$

where we can take $C = 2$ *and* $K = 256$*. Moreover the associated eigenvector is in the space* $\mathbf{E}_2$ *of state* $\mathbf{h}$ *such that* $(\mathbf{h})_N$ *is a fourth degree even polynomial.*

Proof. See section 3.2.

Since μ/ρ is the average number of particle in the steady state, the conditions in (5) are not too restrictive.

It is possible to see that, like in the case of the standard Kac evolution, the L^2 norm discussed above does not scale well with the average number of particles in the system. Since both $\mathscr{T}$ and $\mathscr{K}$ are unbounded operator and they do not commute, we first need to show that (1) defines an evolution in $L^1_s(\mathscr{R})$.

Lemma 3 *The operator* $e^{t\mathscr{L}}$ *is continuous on* $L^1_s(\mathscr{R})$*. Moreover for every* $\mathbf{f}$ *we have*

$$\|e^{t\mathscr{L}}\mathbf{f}\|_1 \le \|\mathbf{f}\|_1 \,.$$

Proof. See section 3.3.

We can now introduce the entropy of a state $\mathbf{f}$ relative to the steady state $\boldsymbol{\Gamma}$ as

$$\mathscr{S}(\mathbf{f}\,|\,\boldsymbol{\Gamma}) = \sum_N a_N \int dv_N h_N(\underline{v}_N) \log h_N(\underline{v}_N) \gamma_N(\underline{v}_N)$$

where, as before, $\mathbf{f} = \boldsymbol{\Gamma}\mathbf{h}$ and a_N and γ_N are defined in (3).

As usual, it is easy to show using convexity that $\mathscr{S}(\mathbf{f}\,|\,\boldsymbol{\Gamma}) \ge 0$, $\mathscr{S}(\mathbf{f}\,|\,\boldsymbol{\Gamma}) = 0$ if and only if $\mathbf{f} = \boldsymbol{\Gamma}$, and $\frac{d}{dt}\mathscr{S}(\mathbf{f}\,|\,\boldsymbol{\Gamma}) \le 0$. In section 3.4, we show that, thanks to the presence of the reservoir, the entropy production rate is strictly negative. More precisely, we obtain

$$\frac{d}{dt}\mathscr{S}(\mathbf{f}(t)\,|\,\boldsymbol{\Gamma}) \le -\rho\mathscr{S}(\mathbf{f}(t)\,|\,\boldsymbol{\Gamma})$$

were $\mathbf{f}(t) = e^{t\mathscr{L}}\mathbf{f}$. The below theorem immediately follows.

Theorem 4 *If* $\mathbf{f}(0)$ *has finite relative entropy*

$$\mathscr{S}(\mathbf{f}(0)\,|\,\boldsymbol{\Gamma}) < \infty$$

then

$$\mathscr{S}(\mathbf{f}(t)\,|\,\boldsymbol{\Gamma}) \le e^{-\rho t}\mathscr{S}(\mathbf{f}(0)\,|\,\boldsymbol{\Gamma}) \,.$$

Proof. See section 3.4.

We can now discuss the validity of a Boltzmann-Kac type equation when the average number of particles in the system goes to infinity. To follow the standard analysis in [11], we have first to define what is a *chaotic sequence* in the present situation. It is natural to consider $\mathbf{f} = (f_1, f_2, \dots)$ a product state if it has the form

$$f_N(\underline{v}_N) = e^{-\eta} \frac{\eta^N}{N!} \prod_{i=1}^{N} g(v_i) \tag{7}$$

where $g(v)$ is a probability density on $\mathbb{R}$ and $\eta > 0$ is the average number of particles. We observe that for the state $\mathbf{f}$ in (7), we have

$$\left(e^{t\mathscr{T}}\mathbf{f}\right)_N = e^{-\eta(t)} \frac{\eta(t)^N}{N!} \prod_{i=1}^{N} g(v_i, t)$$

where $\mathscr{T}$ is defined in (2) and we get

$$\begin{aligned} \dot{\eta}(t) =& \mu - \rho\eta(t) \\ \dot{g}(v,t) =& \frac{\mu}{\eta(t)}(\gamma(v) - g(v,t)) \end{aligned} \tag{8}$$

that is, the thermostat preserves the product structure exactly.

Thus we call a sequence a state $\mathbf{f}_n = (f_{n,1}, f_{n,2}, \dots)$ chaotic if it approaches the structure (7) while the average number of particle $\langle \mathscr{N}\mathbf{f}_n \rangle$ goes to infinity. More precisely, let μ_n be a sequence such that $\lim_{n\to\infty} \mu_n = \infty$ and define

$$F_n^{(k)}(\underline{v}_k) = \left(\frac{\rho}{\mu_n}\right)^k \sum_{N\geq k} \frac{N!}{(N-k)!} \int f_{n,N}(\underline{v}_k, \underline{v}_{N-k}) d\underline{v}_{N-k}\,. \tag{9}$$

We say that $\mathbf{f}_n$ is *chaotic (w.r.t. μ_n)* if

$$\lim_{n\to\infty} F_n^{(1)}(v) = F(v) \tag{10}$$

for some F and

$$\lim_{n\to\infty} F_n^{(k)}(\underline{v}_k) = \prod_{i=1}^{k} F(v_i) \tag{11}$$

where all limits are meant in a weak sense. Observe that

$$\int F(v)dv = \lim_{n\to\infty} \frac{\langle \mathscr{N}\mathbf{f}_n \rangle \rho}{\mu_n}$$

so that we can see $F(v)$ as the *relative particles density*.

Let now

$$\mathbf{f}_n(t) = e^{\mathscr{L}_n t}\mathbf{f}_n(0)$$

where $\mathscr{L}_n$ is given by (1) with $\mu = \mu_n$. In section 3.5, we prove that $e^{\mathscr{L}_n t}$ propagates chaos in the sense that, if $\mathbf{f}_n(0)$ forms a chaotic sequence, then $\mathbf{f}_n(t)$ also forms a chaotic sequence for every t. Thus we get the following theorem.

Theorem 5 *The relative particle density*

$$F(v,t)=\lim_{n\to\infty}\frac{\rho}{\mu_n}\sum_{N=1}^{\infty}N\int f_{n,N}(v,\underline{v}_{N-1},t)d\underline{v}_{N-1}$$

satisfies the Boltzmann-Kac type equation

$$\begin{aligned}\frac{d}{dt}F(v,t) &= -\rho(F(v,t)-\gamma(v))+ \\ &\lambda\int_{\mathbb{R}}dw\int\frac{d\theta}{2\pi}[F(v\cos\theta+w\sin\theta,t)F(-v\sin\theta+w\cos\theta,t)-F(w,t)F_t(w,t)]\,.\end{aligned} \tag{12}$$

We conclude this section looking at the evolution of some thermodynamic quantities of interest. Since the Kac collision operator $\mathcal{K}$ preserves energy and number of particles, one can derive autonomous equations for the evolutions of $N(t)=\langle\mathcal{N}\mathbf{f}(t)\rangle$ and $E(t)=\langle\mathcal{E}\mathbf{f}(t)\rangle$ where $(\mathcal{E}\mathbf{f})_N=\sum_{i=1}^{N}v_i^2 f_N(\underline{v}_N)$. Indeed, we obtain

$$\begin{aligned}\frac{d}{dt}N(t) &= \mu-\rho N(t) \\ \frac{d}{dt}E(t) &= \frac{\mu}{2\pi}-\rho E(t)\,.\end{aligned}$$

Letting $e(t)=E(t)/N(t)$, we get

$$\frac{d}{dt}e(t)=\frac{\mu}{N(t)}\left(\frac{1}{2\pi}-e(t)\right)\,.$$

Observe that $e(t)$ is not the natural definition of average kinetic energy per particle. A more interesting quantity is $e(t)=\langle v_1^2\mathbf{f}\rangle$, but it is not possible to obtain a closed expression for its evolution.

3 Proofs

3.1 Proof of Theorem 1

To prove Theorems 1 and 2, we will construct a basis of eigenvectors for the generator

$$\mathcal{G}=\mu(\mathcal{P}^- -\mathrm{Id})+\rho(\mathcal{P}^+ -\mathcal{N})$$

of the evolution due to the thermostat. Our construction is very similar to the construction of the Fock space for a bosonic field theory.

We start by defining

$$\begin{aligned}(\mathcal{P}^+(g)\mathbf{h})_N(\underline{v}_N) &= \sum_{i=1}^{N}h_{N-1}(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_N)g(v_i) \\ (\mathcal{P}^-(g)\mathbf{h})_N(\underline{v}_N) &= \frac{1}{N+1}\sum_{i=1}^{N+1}\int dwe^{-\pi w^2}g(w)h_{N+1}(\underline{v}_{N,i}(w))\end{aligned}$$

where $\underline{v}_{N,i}(w) = (v_1,\ldots,v_{i-1},w,v_i,\ldots,v_N)$. With this notation, $\mathscr{P}^+$ and $\mathscr{P}^-$ from the introduction are $\mathscr{P}^+(1)$ and $\mathscr{P}^-(1)$, respectively.

Setting $\underline{v}_N^i = (v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_N)$ we get

$$\begin{aligned}\rho(\mathbf{h},\mathscr{P}^+(g)\mathbf{j})_N = \rho a_N \sum_{i=1}^N \int d\underline{v}_N \gamma_N(\underline{v}_N) h_N(\underline{v}_N) j_{N-1}(\underline{v}_N^i) g(v_i) = \\ \rho a_N \sum_{i=1}^N \int d\underline{v}_N^i \gamma_{N-1}(\underline{v}_N^i) \left(\int dv_i e^{-\pi v_i^2} g(v_i) h_N(\underline{v}_N) \right) j_{N-1}(\underline{v}_N^i) = \\ \mu(\mathscr{P}^-(g)\mathbf{h},\mathbf{j}) .\end{aligned} \tag{13}$$

It immediately follows that $\mathscr{G}$ is self adjoint.

Lemma 6 *$\mathscr{G}$ is non positive and $\mathscr{G}\mathbf{h} = 0$ if and only if $\mathbf{h} = c\mathbf{e}^0$, where $\mathbf{e}^0$ is given by $e_N^0(\underline{v}_N) = 1$ for every N and $\underline{v}_N$.*

Proof. From (13) we get $\rho(\mathbf{h},\mathscr{P}^+\mathbf{h}) = \mu(\mathscr{P}^-\mathbf{h},\mathbf{h})$. Moreover, we have

$$\begin{aligned}\rho(\mathbf{h},\mathscr{P}^+\mathbf{h}) = \sum_{N=1}^\infty \left[\sum_{i=1}^N \int d\underline{v}_N \gamma_N(\underline{v}_N) \left(\sqrt{\rho a_N} h_N(\underline{v}_N)\right) \left(\sqrt{\frac{\mu}{N} a_{N-1}} h_{N-1}(\underline{v}_N^i) \right) \right] \le \\ \sum_{N=1}^\infty \sum_{i=1}^N \left[\frac{1}{2} \rho a_N \int d\underline{v}_N \gamma_N(\underline{v}_N) h_N^2 + \frac{1}{2}\frac{\mu}{N} a_{N-1} \int d\underline{v}_N \gamma_N(\underline{v}_N) \left(h_{N-1}(\underline{v}_N^i) \right)^2 \right] = \\ \sum_{N=0}^\infty \left[\frac{1}{2} N \rho a_N \int d\underline{v}_N \gamma_N(\underline{v}_N) h_N^2 + \frac{1}{2}\mu a_N \int d\underline{v}_N \gamma_N(\underline{v}_N) h_N^2 \right] = \frac{1}{2}(\mathbf{h},(\rho\mathscr{N}+\mu)\mathbf{h})\end{aligned}$$

from which non negativity follows immediately. Furthermore, we see that $\mathscr{G}\mathbf{h} = 0$ if and only if:

$$\sqrt{\rho a_N} h_N(\underline{v}_N) = \sqrt{\frac{\mu}{N} a_{N-1}} h_{N-1}(\underline{v}_N^i)$$

or $h_N(\underline{v}_N) = h_{N-1}(\underline{v}_N^i)$ for every i and N which implies that $h_N \equiv h_0$. □

We can now study the commutation relation of the operators $\mathscr{P}^\pm(g)$. Setting $\{\mathscr{A},\mathscr{B}\} = \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}$, we obtain the following Lemma.

Lemma 7 *We have*

$$\begin{aligned}\{\mathscr{P}^+(g_1),\mathscr{P}^-(g_2)\} &= -(g_1,g_2)\mathrm{Id} \\ \{\mathscr{P}^+(g_1),\mathscr{P}^+(g_2)\} = \{\mathscr{P}^-(g_1),\mathscr{P}^-(g_2)\} &= 0 \\ \{\mathscr{N},\mathscr{P}^\pm(g)\} &= \pm\mathscr{P}^\pm(g)\end{aligned}$$

where

$$(g_1,g_2) = \int_{\mathbb{R}} g_1(w) g_2(w) e^{-\pi w^2} dw .$$

Proof. Given $\underline{v} \in \mathbb{R}^N$, let

$$V_{i,j}(\underline{v},w) = \begin{cases} (v_1,\dots,v_{i-1},v_{i+1},\dots,v_{j-1},w,v_j,\dots,v_N), & i<j \\ (v_1,\dots,v_{i-1},w,v_{i+1},\dots,v_N), & i=j \\ (v_1,\dots,v_{j-1},w,v_j,\dots,v_{i-1},v_{i+1},\dots,v_N), & i>j \end{cases}$$

We can now write:

$$\begin{aligned}(\mathscr{P}^+(g_1)\mathscr{P}^-(g_2)\mathbf{h})_N(\underline{v}) &= \frac{1}{N}\sum_{i,j=1}^{N}\int dwe^{-\pi w^2} h_N(V_{i,j}(\underline{v},w))g_1(v_i)g_2(w) \\ &= \sum_{i=1}^{N}\int dwe^{-\pi w^2} h(w,v_1,\dots,v_{i-1},v_{i+1},\dots,v_N)g_1(v_i)g_2(w)\end{aligned} \tag{14}$$

where, in the second identity, we have used the symmetry of h_N.
Analogously, let

$$\widetilde{V}_{i,j}(\underline{v},w) = \begin{cases} (v_1,\dots,v_{i-1},v_{i+1},\dots,v_{j-1},w,v_j,\dots,v_N), & i<j \\ (v_1,\dots,v_N), & i=j \\ (v_1,\dots,v_{j-1},w,v_j,\dots,v_{i-2},v_i,\dots,v_N), & i>j \end{cases}$$

and $\tilde{v}_{i,j}(\underline{v},w) = v_i$ if $i \neq j$ and w otherwise so that

$$\begin{aligned}&(\mathscr{P}^-_N(g_2)\mathscr{P}^+_{N+1}(g_1)\mathbf{h})_N(\underline{v}) = \\ &\frac{1}{N+1}\sum_{i,j=1}^{N+1}\int dwe^{-\pi w^2} h_N(\tilde{V}_{i,j}(\underline{v},w))g_1(\tilde{v}_{i,j}(\underline{v},w))g_2(w) = \\ &\frac{1}{N+1}\sum_{i\neq j}^{N+1}\int dwe^{-\pi w^2} h_N(\tilde{V}_{i,j}(\underline{v},w))g_1(v_i)g_2(w) + \\ &\int dwe^{-\pi w^2} g_1(w)g_2(w)h_N(\underline{v}_N) = \\ &\sum_{i=1}^{N}\int dwe^{-\pi w^2} h(w,v_1,\dots,v_{i-1},v_{i+1},\dots,v_N)g_1(v_i)g_2(w) + \\ &(g_1,g_2)h_N(v_1,\dots,v_N)\end{aligned} \tag{15}$$

where we have again used the symmetry of h_N. Subtracting (15) from (14) gives the first set of relations. The second and third set of relations are straightforward. □

We can now define the analogues of the creation and annihilation operators as

$$\begin{aligned}\mathscr{R}^+(g) &= \sqrt{\frac{\rho}{\mu}}\mathscr{P}^+(g) - \sqrt{\frac{\mu}{\rho}}(g,1) \\ \mathscr{R}^-(g) &= \sqrt{\frac{\mu}{\rho}}\mathscr{P}^-(g) - \sqrt{\frac{\mu}{\rho}}(g,1)\end{aligned}$$

that satisfy $\mathscr{R}^+(g)^* = \mathscr{R}^-(g)$, $\mathscr{R}^-(g)\mathbf{e}^0 = 0$, and

$$\{\mathscr{G},\mathscr{R}^+(g)\} = -\rho\mathscr{R}^+(g)\,, \qquad \{\mathscr{G},\mathscr{R}^-(g)\} = \rho\mathscr{R}^-(g)\,. \tag{16}$$

Since the collision operator $\mathscr{K}$ preserves the space of polynomials of a given degree, we choose as an orthonormal basis for $L^2(\mathbb{R},\gamma(v))$ the polynomials

$$H_n(v) = \frac{1}{\sqrt{n!}}h_n(\sqrt{2\pi}v)\,.$$

where

$$h_n(v) = (-1)^n e^{\frac{v^2}{2}}\frac{d^n}{dv^n}e^{\frac{v^2}{2}}$$

are the standard Hermite polynomials. For every sequence $\underline{\alpha} = (\alpha_1,\alpha_2,\dots)$ such that $\alpha_i \in \mathbb{N}^+$ and $\sum_{i=0}^\infty \alpha_i < \infty$, we define

$$\mathbf{e}_{\underline{\alpha}} = \prod_{i=0}^\infty \frac{(\mathscr{R}_i^+)^{\alpha_i}\mathbf{e}^0}{\sqrt{\alpha_i!}}$$

where $\mathscr{R}_n^\pm = \mathscr{R}^\pm(H_n)$. Observe that for $n \neq m$

$$\{\mathscr{R}_n^\pm,\mathscr{R}_m^\pm\} = 0$$

while

$$\{\mathscr{R}_n^+,\mathscr{R}_n^-\} = -\mathrm{Id}\,.$$

Lemma 8 *The vectors* $\mathbf{e}_{\underline{\alpha}}$ *with* $\lambda(\underline{\alpha}) := \sum_{i=0}^\infty \alpha_i < \infty$ *form an orthonormal basis in* $L^2_s(\mathscr{R},\boldsymbol{\Gamma})$*. Moreover, we have*

$$\mathscr{G}\mathbf{e}_{\underline{\alpha}} = -\rho\lambda(\underline{\alpha})\mathbf{e}_{\underline{\alpha}}\,. \tag{17}$$

Proof. If $n_1 \neq n_2$ we have

$$((\mathscr{R}_{n_1}^+)^{\alpha_1}\mathbf{e}^0,(\mathscr{R}_{n_2}^+)^{\alpha_2}\mathbf{e}^0) = 0$$

while

$$((\mathscr{R}_n^+)^{\alpha_1}\mathbf{e}^0,(\mathscr{R}_n^+)^{\alpha_2}\mathbf{e}^0) = \alpha_2((\mathscr{R}_n^+)^{\alpha_1-1}\mathbf{e}^0,(\mathscr{R}_n^+)^{\alpha_2-1}\mathbf{e}^0)\,.$$

Assuming $\alpha_1 < \alpha_2$ we get

$$((\mathscr{R}_n^+)^{\alpha_1}\mathbf{e}^0,(\mathscr{R}_n^+)^{\alpha_2}\mathbf{e}^0) = \alpha_1!(\mathbf{e}^0,(\mathscr{R}_n^+)^{\alpha_2-\alpha_1}\mathbf{e}^0)$$

so that

$$((\mathscr{R}_n^+)^{\alpha_1}\mathbf{e}^0,(\mathscr{R}_n^+)^{\alpha_2}\mathbf{e}^0) = \alpha_1!\delta_{\alpha_1,\alpha_2}$$

from which orthonormality follows easily. Moreover, we can write

$$\mathbf{n} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}(\mathscr{P}^+(1))^n\mathbf{e}^0$$

where $\mathbf{n} = (1,0,0,\dots)$. Thus $\mathbf{n}$ is in the span of the $\mathbf{e}_{\underline{\alpha}}$. Acting with the $\mathscr{P}_n^+$ on $\mathbf{n}$, we can generate the basis of multivariate Hermite polynomials for $L^2_s(\mathbb{R}^N,\gamma_N(\underline{v}_N)d\underline{v}_N)$. Thus the span of the $\mathbf{e}_{\underline{\alpha}}$ contains a basis for each of the $L^2_s(\mathbb{R}^N,\gamma_N(\underline{v}_N)d\underline{v}_N)$. This proves completeness. Equation (17) follows easily from (16). □

Observe that $\mathscr{G}\mathbf{h} = -\rho\mathbf{h}$ if and only if $\mathbf{h} = \sum_i a_i\mathscr{R}_i^+\mathbf{e}^0$. On the other hand we have that $(\mathscr{R}_i^+\mathbf{e}^0,\mathscr{K}\mathscr{R}_i^+\mathbf{e}^0) < 0$ if $i \neq 0,2$ so that $\widetilde{\mathscr{L}} = -\rho\mathbf{h}$ if and only if $\mathbf{h} = a_0\mathscr{R}_0\mathbf{e}^0 + a_1\mathscr{R}_2\mathbf{e}^0$. This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

To prove Theorem 2, we need more information on the action of $\mathscr{K}$ on the vectors $\mathbf{e}_{\underline{\alpha}}$. We start by observing that, from Lemma 12 in [2], we have

$$R_{ij}h_{2n}(v_i) = \tau_n \sum_{k=0}^{n} \binom{n}{k} h_{2k}(v_i)h_{2(n-k)}(v_j)$$

where

$$\tau_n = \int \cos^{2n} \frac{d\theta}{2\pi} = \frac{1}{4^n}\binom{2n}{n}$$

so that we obtain

$$R_{ij}H_{2n}(v_i) = \tau_n \sum_{k=0}^{n} \binom{n}{k} \frac{\sqrt{(2k)![2(n-k)]!}}{\sqrt{(2n)!}} H_{2k}(v_i)H_{2(n-k)}(v_j)\,.$$

Using the above, we can write

$$(\mathscr{K}\mathscr{R}_{2n}\mathbf{e}^0)_N = \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} (N-1)(2\tau_n - 1)\sum_{i=1}^{N} H_{2n}(v_i) + \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} \sum_{k=1}^{n-1}\sum_{i\neq j} \sigma_{n,k} H_{2k}(v_i)H_{2(n-k)}(v_j)$$

where

$$\sigma_{n,k} = \tau_n \frac{\binom{n}{k}}{\sqrt{\binom{2n}{2k}}} = \sqrt{\tau_n\tau_k\tau_{n-k}}\,. \tag{18}$$

This gives us

$$\begin{aligned}\mathscr{K}\mathscr{R}_{2n}\mathbf{e}^0 =& (2\tau_n-1)\mathscr{R}_{2n}\mathscr{N}\mathbf{e}^0 + \sqrt{\frac{\mu}{\rho}}\sum_{k=1}^{n-1}\sigma_{n,k}\mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0 = \\ & \frac{\mu}{\rho}(2\tau_n-1)\mathscr{R}_{2n}\mathbf{e}^0 + \sqrt{\frac{\mu}{\rho}}(2\tau_n-1)\mathscr{R}_0\mathscr{R}_{2n}\mathbf{e}^0 + \\ & \sqrt{\frac{\mu}{\rho}}\sum_{k=1}^{n-1}\sigma_{n,k}\mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0\end{aligned}$$

where we have used that $\mathscr{N}\mathbf{e}^0 = \sqrt{\frac{\mu}{\rho}}\mathscr{R}_0\mathbf{e}^0 + \frac{\mu}{\rho}\mathbf{e}^0$.

Because $h_{2n+1}(v)$ is an odd function of v, we get

$$R_{ij}h_{2n+1}(v_i) = 0$$

that gives

$$\mathscr{K}\mathscr{R}_{2n+1}\mathbf{e}^0 = -\frac{\mu}{\rho}\mathscr{R}_{2n+1}\mathbf{e}^0 - \sqrt{\frac{\mu}{\rho}}\mathscr{R}_0\mathscr{R}_{2n+1}\mathbf{e}^0\,.$$

Let now $\mathbf{V}_m = \mathrm{span}\{\mathbf{e}_{\underline{\alpha}} | \sum_{i=0}^{\infty} i\alpha_i = m\}$ and

$$\delta_m = \inf_{\substack{\mathbf{h}\in\mathbf{V}_m \\ \|\mathbf{h}\|=1,\, \mathbf{h}\perp\mathbf{E}_1\oplus\mathbf{E}_0}} (\mathbf{h}, -\mathscr{L}\mathbf{h}).$$

Since $\mathscr{K}\mathbf{V}_m = \mathbf{V}_m$ we have that $\Delta = -\inf_m \delta_m$.

Observe that $\mathbf{V}_0 = \mathrm{span}\{\mathscr{R}_0^n\mathbf{e}^0\}$ and $\mathbf{V}_2 = \mathrm{span}\{\mathscr{R}_0^n\mathscr{R}_2\mathbf{e}^0, \mathscr{R}_0^n\mathscr{R}_1^2\mathbf{e}^0\}$ so that $\delta_0 = \delta_2 = 2\rho$ while for $m \neq 0,2$, $\mathbf{V}_m \perp \mathbf{E}_1 \oplus \mathbf{E}_0$. Thus we need a lower bound for δ_m for m odd and for m even greater than 2.

If $m = 2n+1$, we can write any $\mathbf{h} \in \mathbf{V}_m$ as

$$\mathbf{h} = a\mathscr{R}_{2n+1}\mathbf{e}^0 + b\mathscr{R}_0\mathscr{R}_{2n+1}\mathbf{e}^0 + \mathbf{j} = a\mathscr{R}_{2n+1}\mathbf{e}^0 + \mathbf{k}$$

with $\mathbf{j} \perp \mathscr{R}_{2n+1}\mathbf{e}^0$ and $\mathbf{j} \perp \mathscr{R}_0\mathscr{R}_{2n+1}\mathbf{e}^0$ and we obtain

$$\begin{aligned}
(\mathbf{h}, -\mathscr{L}\mathbf{h}) =& (\mathbf{h}, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{h}) = \\
& a^2(\lambda+\rho) + (\mathbf{k}, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{k}) + 2a(\mathscr{R}_{2n+1}\mathbf{e}^0, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{k}) \geq \\
& a^2(\lambda+\rho) + 2\rho(b^2 + \|\mathbf{j}\|^2) - 2\lambda|ab|\sqrt{\frac{\rho}{\mu}} \geq \\
& a^2\left(\rho + \lambda - \lambda\sqrt{\frac{\rho}{\mu}}\right) + b^2\left(2\rho - \lambda\sqrt{\frac{\rho}{\mu}}\right) + 2\rho\|\mathbf{j}\|^2.
\end{aligned}$$

Since $\|\mathbf{h}\|^2 = a^2 + b_0^2 + \|\mathbf{j}\|^2$ we have

$$\delta_{2n+1} \geq \min\left(\rho + \lambda - \lambda\sqrt{\frac{\rho}{\mu}}, 2\rho - \lambda\sqrt{\frac{\rho}{\mu}}\right).$$

Similarly, every $\mathbf{h} \in \mathbf{V}_{2n}$ with $n \geq 2$ can be written as

$$\mathbf{h} = a\mathscr{R}_{2n}\mathbf{e}^0 + \sum_{k=0}^{n/2} b_k\mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0 + \mathbf{j} = a\mathscr{R}_{2n}\mathbf{e}^0 + \mathbf{k}$$

where $\mathbf{j} \perp \mathscr{R}_{2n}\mathbf{e}^0$ and $\mathbf{j} \perp \mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0$. Again we obtain

$$\begin{aligned}
(\mathbf{h}, -\mathscr{L}\mathbf{h}) &= (\mathbf{h}, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{h}) = \\
& a^2((1-2\tau_n)\lambda + \rho) + (\mathbf{k}, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{k}) + 2a(\mathscr{R}_{2n}\mathbf{e}^0, (-\mathscr{G} - \lambda\frac{\rho}{\mu}\mathscr{K})\mathbf{k}) \geq \\
& a^2((1-2\tau_n)\lambda + \rho) + 2\rho\left(\sum_{k=0}^{n/2}\varepsilon_{n,k}b_k^2 + \|\mathbf{j}\|^2\right) \\
&\qquad - 4\lambda\sqrt{\frac{\rho}{\mu}}\sum_{k=1}^{n/2}|ab_k|\sigma_{n,k} - 2\lambda|ab_0|\sqrt{\frac{\rho}{\mu}}(1-2\tau_n)
\end{aligned}$$

where $\varepsilon_{n,k} = 2$ if $k = n-k$ and 1 otherwise. Note that the first term in the last line is due to the fact that, if $k \neq n-k$ in the scalar product of $\sum_{k=0}^{n/2} b_k \mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0$ and $\sum_{k=1}^{n-1} \sigma_{n,k}\mathscr{R}_{2k}\mathscr{R}_{2(n-k)}\mathbf{e}^0$, the term $b_k\sigma_{n,k}$ appears twice since $\sigma_{n,k} = \sigma_{n,n-k}$. Alternatively, if $k = n-k$, it appears only once, but the scalar product is 2.

As before, the above gives

$$
\begin{aligned}
(\mathbf{h}, -\mathscr{L}\mathbf{h}) \geq a^2((1-2\tau_n)\lambda + \rho) + 2\rho \left(\sum_{k=0}^{n/2} \varepsilon_{n,k} b_k^2 + \|\mathbf{j}\|^2 \right) - \\
\lambda \sqrt{\frac{\rho}{\mu}} \left[2\sum_{k=1}^{n/2} a^2 \sigma_{n,k}^2 + a^2(1-2\tau_n)^2 + 2\sum_{k=0}^{n/2} b_k^2 \right] \geq \\
\left(2\rho - 2\lambda\sqrt{\frac{\rho}{\mu}}\right) \left[\sum_{k=0}^{n/2} \varepsilon_{n,k} b_k^2\right] + a^2 \left((1-2\tau_n)\lambda + \rho - \lambda\sqrt{\frac{\rho}{\mu}} A_{2n} \right) + 2\rho\|\mathbf{j}\|^2
\end{aligned}
$$

where

$$
A_{2n} = (1-2\tau_n)^2 + 2\sum_{k=1}^{n/2} \sigma_{n,k}^2 \,.
$$

Observe now that

$$
\log \tau_n = \log \prod_{i=1}^{n} \left(1 - \frac{1}{2i}\right) \leq -\sum_{i=1}^{n} \frac{1}{2i} \leq -\frac{1}{2}\log n \quad \Rightarrow \quad \tau_n \leq \frac{1}{\sqrt{n}}
$$

and, from (18), we have

$$
\sigma_{n,k}^2 \leq \tau_n \frac{1}{n} \frac{1}{\sqrt{\frac{k}{n}}\sqrt{1-\frac{k}{n}}}
$$

so that

$$
\sum_{k=1}^{n/2} \sigma_{n,k}^2 \leq \tau_n \int_0^{\frac{1}{2}} \frac{1}{\sqrt{x(1-x)}} dx = \frac{\pi}{2}\tau_n
$$

and thus $A_{2n} \leq 2$ for every n. Finally, we get

$$
\delta_{2n} \geq \min\left(2\rho - 2\lambda\sqrt{\frac{\rho}{\mu}}, (1-2\tau_n)\lambda + \rho - 2\lambda\sqrt{\frac{\rho}{\mu}}\right) .
$$

Moreover, because $(\mathscr{R}_4\mathbf{e}^0, -\mathscr{L}\mathscr{R}_4\mathbf{e}^0) = \rho + \lambda/4$, we get $\delta_4 \leq \rho + \lambda/4$. Since

$$
\begin{aligned}
\min\left(2\rho - \lambda\sqrt{\frac{\rho}{\mu}}, \rho + \lambda - \lambda\sqrt{\frac{\rho}{\mu}}\right) \geq \\
\min\left(2\rho - 2\lambda\sqrt{\frac{\rho}{\mu}}, (1-2\tau_3)\lambda + \rho - 2\lambda\sqrt{\frac{\rho}{\mu}}\right)
\end{aligned}
$$

for (6) to be valid, we need

$$\min\left(2\rho-2\lambda\sqrt{\frac{\rho}{\mu}},(1-2\tau_3)\lambda+\rho-2\lambda\sqrt{\frac{\rho}{\mu}}\right)>\rho+\frac{\lambda}{4}$$

which requires

$$(1-2\tau_3)\lambda+\rho-2\lambda\sqrt{\frac{\rho}{\mu}}>\rho+\frac{\lambda}{4}\quad\Rightarrow\quad\frac{\mu}{\rho}>256$$
$$2\rho-2\lambda\sqrt{\frac{\rho}{\mu}}>\rho+\frac{\lambda}{4}\quad\Rightarrow\quad\rho>\frac{\lambda}{4}+2\lambda\sqrt{\frac{\rho}{\mu}}\,.$$

This concludes the proof of Theorem 2. □

3.3 Proof of Lemma 3.

Clearly $\|e^{t\tilde{\lambda}\mathscr{K}}\|\le 1$ since $\|e^{t\tilde{\lambda}K_N}\|\le 1$ for every N. We can now use a Duhamel style expansion to obtain

$$e^{t(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N}+\rho\mathscr{O})}=e^{(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t}+ \\ e^{(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t}\sum_{n\ge 1}\int_{0<t_1<\cdots<t_n<t}\prod_{i=1}^{n}e^{-(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t_i}\rho\mathscr{O}e^{(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t_i}dt_1\cdots dt_n\,. \tag{19}$$

Observing that

$$\left\|\left(e^{(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t}\int_{0<t_1<\cdots<t_n<t}\prod_{i=1}^{n}e^{-(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t_i}\rho\mathscr{O}e^{(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N})t_i}dt_1\cdots dt_n\mathbf{f}\right)_N\right\|_1\le$$
$$e^{-\rho Nt}\frac{(N+n)!}{N!}\int_{0<t_1<\cdots<t_n<t}\prod_{i=1}^{n}\int d\underline{v}_N$$
$$\left|e^{-(\tilde{\lambda}K_{N+i-1}-\rho(N+i-1))t_i}\rho\int dv_{N+i}e^{(\tilde{\lambda}K_{N+i}-\rho(N+i))t_i}dt_1\cdots dt_nf_{N+n}(\underline{v}_{N+n})\right|\le$$
$$e^{-\rho Nt}\frac{(N+n)!}{N!}\int_{0<t_1<\cdots<t_n<t}\prod_{i=1}^{n}e^{\rho(N+i-1)t_i}\rho e^{-\rho(N+i)t_i}dt_1\cdots dt_n\|f_{N+n}\|_1=$$
$$\binom{N+n}{N}e^{-\rho Nt}\left(1-e^{-\rho t}\right)^n\|f_{N+n}\|_1$$

we get

$$\|e^{t(\tilde{\lambda}\mathscr{K}-\rho\mathscr{N}+\rho\mathscr{O})}\mathbf{f}\|_1\le\sum_{N=0}^{\infty}\sum_{n=0}^{\infty}\binom{N+n}{N}e^{-\rho Nt}\left(1-e^{-\rho t}\right)^n\|f_{N+n}\|_1=\sum_{N=0}^{\infty}\|f_N\|_1$$

and so that $\|e^{t\tilde{\lambda}\mathscr{K}-\rho\mathscr{N}+\rho\mathscr{O}}\|\le 1$. Since $\|\mathscr{I}\|_1\le 1$ it is easy to repeat an argument like (19) and conclude the proof of Lemma 3. □

3.4 Proof of theorem 4.

We first need an explicit expression for $\frac{d}{dt}\mathscr{S}(\mathbf{f}(t)\,|\,\boldsymbol{\Gamma})$. To simplify notations, given $\mathbf{f}=\boldsymbol{\Gamma}\mathbf{h}$, we set $S_t(\mathbf{h})=\mathscr{S}(\mathbf{f}(t)\,|\,\boldsymbol{\Gamma})$.

Lemma 9 *Let* $\mathbf{h}=(h_1,h_2,\ldots)$. *We have*

$$\dot{S}_t(\mathbf{h})\leq -\mu\sum_{N=0}^{\infty}a_N\int d\underline{v}_{N+1}(h_{N+1}-h_N)(\log h_{N+1}-\log h_N)\gamma_{N+1}(\underline{v}_{N+1})$$

Proof. We can write:

$$\begin{aligned}\dot{S}_t(\mathbf{h})=&\sum_{N=0}^{\infty}a_N\int d\underline{v}_N\gamma_N\dot{h}_N\log h_N=\\ &\sum_{N=0}^{\infty}a_N\int d\underline{v}_N\gamma_N\left(\rho(\mathscr{P}^+\mathbf{h})_N+\mu(\mathscr{P}^-\mathbf{h})_N-((\mu+\rho N)\mathbf{h})_N+K_Nh_N\right)\log h_N\leq\\ &\sum_{N=0}^{\infty}a_N\int d\underline{v}_N\gamma_N\left(\rho(\mathscr{P}^+\mathbf{h})_N+\mu(\mathscr{P}^-\mathbf{h})_N-((\mu+\rho N)\mathbf{h})_N\right)\log h_N\end{aligned}$$

Observe that

$$\begin{aligned}\int d\underline{v}_N\gamma_N\rho(\mathscr{P}^+\mathbf{h})_N\log h_N&=\int d\underline{v}_N\gamma_NN\rho h_{N-1}\log h_N\\ \int d\underline{v}_N\gamma_N\mu(\mathscr{P}^-\mathbf{h})_N\log h_N&=\int d\underline{v}_{N+1}\gamma_{N+1}\mu h_{N+1}\log h_N\end{aligned}$$

from which we get

$$\begin{aligned}\dot{S}_t(\mathbf{h})\leq&\sum_{N=1}a_N\int d\underline{v}_N\gamma_NN\rho(h_{N-1}-h_N)\log h_N+\\ &\mu\sum_{N=0}a_N\int d\underline{v}_{N+1}\gamma_{N+1}(h_{N+1}-h_N)\log h_N\end{aligned}$$

The thesis follows by reindexing the first sum and observing that $\rho Na_n=\mu a_{N-1}$. □

Given $\mathbf{f}=\boldsymbol{\Gamma}\mathbf{h}\in\mathscr{L}^1_s(\mathscr{R})$ such that $S(\mathbf{h})<\infty$ we set

$$\begin{aligned}\Psi(\mathbf{h})&=\sum_{N=0}^{\infty}a_N\int d\underline{v}_{N+1}(h_{N+1}-h_N)(\log h_{N+1}-\log h_N)\gamma_{N+1}(\underline{v}_{N+1})\\ E(\mathbf{h})&=\sum_{N=0}^{\infty}a_N\int d\underline{v}_Nh_N(\underline{v}_N)\gamma_N(\underline{v}_N).\end{aligned}$$

Lemma 10 *If* $\boldsymbol{\Gamma}\mathbf{h}\in L^1_s(\mathscr{R})$ *with* $S(\mathbf{h})<\infty$, *then*

$$S(\mathbf{h})\leq E(\mathbf{h})\log E(\mathbf{h})+\frac{\mu}{\rho}\Psi(\mathbf{h})\,. \tag{20}$$

If $E(\mathbf{h})=1$ *then*

$$\dot{S}(\mathbf{h})\leq-\rho S(\mathbf{h}).$$

The idea of the proof of (20) is to think of the entry and exit process defined by the thermostat as a continuous family of independent entry processes, one for each possible velocity v, with entry rates $\mu\gamma(v)dv$, while each particle in the system leaves with rate ρ independent of its velocity. Clearly such a description makes little mathematical sense so that, as first step, we would like to restrict the velocity of each particle to assume only a finite number of values v_k, $k = 1,\dots,K$, characterized by entry rates ω_k. A better implementation of this consists of approximating each h_N by a suitably chosen simple functions. After this, using convexity, we reduce the proof of (20) to the case with $K = 1$, essentially equivalent to the case in which all particles in the thermostat have the same velocity and thus velocity is irrelevant. In this situation, we further approximate the infinite reservoir by a large finite reservoir containing N particles that enter and leave the system, independently from each other, at a suitable rate. Convexity will allow us to reduce this situation to that of a single particle jumping from the system to the reservoir. The final step is thus Lemma 12 below that deals with this situation. This argument is inspired by the proof of the Logarithmic Sobolev Inequality in [7].

Proof of Lemma 10. A way to make the first step of the discussion above rigorous is to approximate each h_N by a simple function on a partition of $\mathbb{R}^N$ made by rectangles obtained as the Cartesian product of a finite measurable partition of $\mathbb{R}$.

More precisely, given $\mathscr{B} = \{B_k\}_{k=1}^K$ with $B_k \subset \mathbb{R}$ measurable and that $\bigcup_k B_k = \mathbb{R}$ and $B_k \cap B_{k'} = \emptyset$ if $k \neq k'$, call $B_{\underline{k}} = \times_i B_{k_i} \subset \mathbb{R}^N$ where $\underline{k} = (k_1,\dots,k_N) \in \{1,\dots,K\}^N$. If $\underline{v} \in B_{\underline{k}}$, define

$$h_{N,\mathscr{B}}(\underline{v}_N) = \bar{h}_N(\underline{k})$$

where

$$\bar{h}_N(\underline{k}) = \frac{1}{\omega_{\underline{k}}} \int_{B_{\underline{k}}} h_N(\underline{v}_N)\gamma_N(\underline{v}_N)d\underline{v}_N$$

with $\omega_{\underline{k}} = \prod_i \omega_{k_i}$ and

$$\omega_k = \int_{B_k} \gamma(v)dv\,.$$

For every ε we can find $\mathscr{B}$ such that

$$|S(\mathbf{h}) - S(\mathbf{h}_{\mathscr{B}})| < \varepsilon$$

while, due to convexity,

$$\Psi(\mathbf{h}_{\mathscr{B}}) \leq \Psi(\mathbf{h}) \qquad E(\mathbf{h}_{\mathscr{B}}) = E(\mathbf{h})\,.$$

Since h_N is invariant under permutation of its arguments, it follows that $\bar{h}_N(\underline{k})$ depends only on the number n_q of $k_i = q$, that is the number of particles whose velocity is in B_q. Thus, we can describe the state of the system by counting the number of particles with velocity in each set of the partition $\mathscr{B}$.

Let's define $\underline{n}(\underline{k}) = (n_1(\underline{k}),\dots,n_K(\underline{k})) \in \mathbb{N}^K$ as

$$n_q(\underline{k}) = \sum_i \delta_{q,k_i}\,.$$

Because of the permutation symmetry of $\mathbf{h}$, the function $F:\mathbb{N}^K\to\mathbb{R}$ given by

$$F(\underline{n})=\bar{h}_N(\underline{k})\quad\text{if}\quad \underline{n}=\underline{n}(\underline{k}),\ \text{ and } N=\sum_{k=1}^K n_k:=|\underline{n}|$$

is well defined. We can now write

$$\begin{aligned}E(\mathbf{h}_{\mathscr{B}})=&\sum_N a_N\sum_{\underline{k}\in\{1,\ldots,K\}^N}\bar{h}_N(\underline{k})\omega_{\underline{k}}=\\&\sum_N\frac{e^{\frac{\mu}{\rho}}}{N!}\left(\frac{\mu}{\rho}\right)^N\sum_{|\underline{n}|=N}\binom{N}{n_1,\ldots,n_K}F(\underline{n})\prod_{k=1}^K\omega_k^{n_k}=\\&\sum_{\underline{n}}F(\underline{n})\prod_{k=1}^K\pi_{\alpha_k}(n_k):=\widetilde{E}_K(F)\end{aligned}$$

where $\alpha_k=\mu\omega_k/\rho$ and

$$\pi_\alpha(n)=e^{-\alpha}\frac{\alpha^n}{n!},$$

that is π_{α_k} is a Poisson distribution with expected value $\alpha_k=\mu\omega_k/\rho$. Similarly we have

$$\begin{aligned}S(\mathbf{h}_{\mathscr{B}})=&\sum_N a_N\sum_{\underline{k}\in\{1,\ldots,K\}^N}\bar{h}(\underline{k})\log(\bar{h}(\underline{k}))\omega_{\underline{k}}=\\&\sum_{\underline{n}\in\mathbb{N}^K}F(\underline{n})\log(F(\underline{n}))\prod_{k=1}^K\pi_{\alpha_k}(n_k):=\widetilde{S}_K(F)\end{aligned}$$

Finally setting $\underline{n}^q=(n_1,\ldots,n_q+1,\ldots,n_K)$ we get

$$\begin{aligned}&\Psi(\mathbf{h}_{\mathscr{B}})=\\&\sum_N a_N\sum_{\underline{k}\in\{1,\ldots,K\}^N}\sum_q(\bar{h}_{N+1}(\underline{k},q)-\bar{h}_N(\underline{k}))(\log\bar{h}_{N+1}(\underline{k},q)-\log\bar{h}_{N-1}(\underline{k}))\omega_{\underline{k}}\omega_q=\\&\frac{\rho}{\mu}\sum_q\sum_{\underline{n}}\alpha_q\left(F(\underline{n}^q)-F(\underline{n})\right)\left(\log F(\underline{n}^q)-\log F(\underline{n})\right)\prod_{k=1}^K\pi_{\alpha_k}(n_k):=\frac{\rho}{\mu}\widetilde{\Psi}_K(F)\end{aligned}$$

so that, to prove (20), we need to show that, for every α_k, $k=1,\ldots,K$, we have

$$\widetilde{S}_K(F)\le\widetilde{\Psi}_K(F)+\widetilde{E}_K(F)\log\widetilde{E}_K(F)\,. \tag{21}$$

Assume now that (21) is valid for all $L<K$. Calling $F_1(n_K)=\widetilde{E}_{K-1}(F(\cdot,n_K))$, we obtain

$$\begin{aligned}\widetilde{S}_K(F)=&\sum_{n_K}\widetilde{S}_{K-1}(F(\cdot,n_K))\pi_{\alpha_K}(n_K)\le\\&\sum_{n_K}\widetilde{\Psi}_{K-1}(F(\cdot,n_K))\pi_{\alpha_K}(n_K)+\sum_{n_K}\widetilde{E}_{K-1}(F(\cdot,n_K))\log\widetilde{E}_{K-1}(F(\cdot,n_K))\pi_{\alpha_K}(n_K)\le\\&\sum_{n_K}\widetilde{\Psi}_{K-1}(F(\cdot,n_K))\pi_{\alpha_K}(n_K)+\widetilde{\Psi}_1(F_1)+\widetilde{E}_1(F_1)\log\widetilde{E}_1(F_1)\end{aligned}$$

where we have used the inductive hypothesis twice. Observing that $\widetilde{E}_1(F_1) = \widetilde{E}_K(F)$ and that, by convexity,

$$\widetilde{\Psi}_1(F_1) = \alpha_K \sum_n (F_1(n+1) - F_1(n))(\log F_1(n+1) - \log F_1(n))\pi_{\alpha_K}(n) \leq$$
$$\alpha_K \sum_{\underline{n}} \left(F(\underline{n}^K) - F(\underline{n})\right)\left(\log F(\underline{n}^K) - \log F(\underline{n})\right)\prod_{k=1}^K \pi_{\alpha_k}(n_k)\,.$$

we get (21) for K. Thus, by induction, to prove (21) for every K we just need to prove it for $K = 1$. This is the content of the following Lemma.

Lemma 11 *Let π_α the Poisson distribution on $\mathbb{N}$ with expected value $\alpha > 0$ and $f : \mathbb{N} \to \mathbb{R}$ be such that*

$$\sum_n f(n)\log f(n)\pi_\alpha(n) < \infty\,,$$

then we have

$$\sum_n f(n)\log f(n)\pi_\alpha(n) \leq \left(\sum_n f(n)\pi_\alpha(n)\right)\log\left(\sum_n f(n)\pi_\alpha(n)\right) +$$
$$\alpha\sum_n \big(f(n+1) - f(n)\big)\big(\log f(n+1) - \log f(n)\big)\,\pi_\alpha(n)\,.$$

Proof. Observe first that

$$\alpha\sum_n \big(f(n+1) - f(n)\big)\big(\log f(n+1) - \log f(n)\big)\,\pi_\alpha(n) =$$
$$\sum_n n\big(f(n) - f(n-1)\big)\big(\log f(n) - \log f(n-1)\big)\,\pi_\alpha(n)$$

and let $\pi_{\alpha,N}(n)$ be the binomial distribution with parameters N and α/N, that is

$$\pi_{\alpha,N}(n) = \binom{N}{n}\left(\frac{\alpha}{N}\right)^n\left(1 - \frac{\alpha}{N}\right)^{N-n}$$

We will prove by induction that

$$\sum_n f(n)\log f(n)\pi_{\alpha,N}(n) \leq \left(\sum_n f(n)\pi_{\alpha,N}(n)\right)\log\left(\sum_n f(n)\pi_{\alpha,N}(n)\right) + \tag{22}$$
$$\sum_n n\big(f(n) - f(n-1)\big)\big(\log f(n) - \log f(n-1)\big)\,\pi_{\alpha,N}(n)$$

and the statement will follow by letting $N \to \infty$.

The base case $N = 1$ is covered by the following Lemma.

Lemma 12 *Let $\mu_x \geq 0$, $x \in \{0,1\}$, be such that $\mu_0 + \mu_1 = 1$ then for every function $f : \{0,1\} \to \mathbb{R}^+$ we have*

$$\sum_{x=0,1} f(x)\log f(x)\mu_x \leq \left(\sum_{x=0,1} f(x)\mu_x\right)\log\left(\sum_{x=0,1} f(x)\mu_x\right) +$$
$$\mu_0\mu_1\big(f(1) - f(0)\big)\big(\log f(1) - \log f(0)\big)\,. \tag{23}$$

Proof. Calling $h(0) = f(0)/(\mu_0 f(0)+\mu_1 f(1))$ and $h(1) = f(1)/(\mu_0 f(0)+\mu_1 f(1))$, (23) becomes

$$\sum_{x=0,1} h(x)\log h(x)\mu_x \le \mu_0\mu_1 \big(h(1)-h(0)\big)\big(\log h(1)-\log h(0)\big)\ .$$

Since $\mu_0 h(0)+\mu_1 h(1) = 1$ we can write $h(0) = 1+\delta\mu_1$ and $h(1) = 1-\delta\mu_0$ and we get

$$\begin{aligned}&\sum_{x=0,1} h(x)\log h(x)\mu_x =\\ &\quad \mu_0\mu_1\delta(\log(1+\delta\mu_1)-\log(1-\delta\mu_0))+\mu_0\log(1+\delta\mu_1)+\mu_1\log(1-\delta\mu_0)\le\\ &\quad \mu_0\mu_1\delta(\log(1+\delta\mu_1)-\log(1-\delta\mu_0)) =\\ &\quad \mu_0\mu_1\big(h(1)-h(0)\big)\big(\log h(1)-\log h(0)\big)\end{aligned}$$

where we have used the concavity of the logarithm. □

Assume now that (22) holds for all $L < N$, call $\beta = (N-1)\alpha/N$, $\mu_0 = 1-\alpha/N$, $\mu_1 = \alpha/N$, and observe that

$$\sum_n f(n)\pi_{\alpha,N}(n) = \sum_x\sum_n f(n+x)\pi_{\beta,N-1}(n)\mu_x\,.$$

Calling

$$\bar f(x) = \sum_n f(n+x)\pi_{\beta,N-1}(n)$$

we get

$$\begin{aligned}&\sum_n f(n)\log f(n)\pi_{\alpha,N}(n) = \sum_x\sum_n f(n+x)\log f(n+x)\pi_{\beta,N-1}(n)\mu_x \le\\ &\quad \sum_x \bar f(x)\log\bar f(x)\mu_x + \sum_x\sum_n n\big(f(n+x)-f(n-1+x)\big)\cdot\\ &\qquad\qquad \big(\log f(n+x)-\log f(n-1+x)\big)\pi_{\beta,N-1}(n)\mu_x \le\\ &\quad \left(\sum_n f(n)\pi_{\alpha,N}(n)\right)\log\left(\sum_n f(n)\pi_{\alpha,N}(n)\right)+ \qquad (24)\\ &\quad \mu_0\mu_1(\bar f(1)-\bar f(0))(\log\bar f(1)-\log\bar f(0)) + \sum_x\sum_n n\big(f(n+x)-f(n-1+x)\big)\cdot\\ &\qquad\qquad \big(\log f(n+x)-\log f(n-1+x)\big)\pi_{\beta,N-1}(n)\mu_x\end{aligned}$$

Observing that

$$\begin{aligned}&\mu_0\mu_1(\bar f(1)-\bar f(0))(\log\bar f(1)-\log\bar f(0))\le\\ &\quad \mu_1\sum_n (f(n+1)-f(n))(\log f(n+1)-\log f(n))\pi_{\beta,N-1}(n) =\\ &\quad \sum_x\sum_n x(f(n+x)-f(n-1+x))(\log f(n+x)-\log f(n-1+x))\pi_{\beta,N-1}(n)\mu_x\end{aligned}$$

and inserting in (24) we get the thesis. □

3.5 Proof of Theorem 5

Given a continuous compact support test function $\phi : \mathbb{R}^k \to \mathbb{R}$, symmetric with respect to the permutation of its variables, we define

$$(\mathbf{f}_n, \phi)_{k,n} = \left(\frac{\rho}{\mu_n}\right)^k \sum_{N\geq k} \frac{N!}{(N-k)!} \int_{\mathbb{R}^N} f_{n,N}(\underline{v}_N)\phi_k(\underline{v}_k)d\underline{v}_N\,.$$

We first extend the analysis in [11] to show that $e^{\tilde{\lambda}\mathscr{K}t}$ propagates chaos in the sense of (9)-(11). More precisely we show that if $\mathbf{f}_n$ forms a chaotic sequence and $\phi : \mathbb{R}^k \to \mathbb{R}$ is a test function then

$$\lim_{n\to\infty}(e^{\tilde{\lambda}_n\mathscr{K}t}\mathbf{f}_n, \phi^{\otimes k})_{k,n} = \left(\int_{\mathbb{R}} \widetilde{F}(v,t)\phi(v)dv\right)^k \tag{25}$$

where $\tilde{\lambda}_n = \lambda\rho/\mu_n$ and $\widetilde{F}_t$ solve the Boltzmann-Kac equation

$$\frac{d}{dt}\widetilde{F}(v,t) = \lambda\int_{\mathbb{R}} dw \int \frac{d\theta}{2\pi}[\widetilde{F}(v\cos\theta + w\sin\theta,t)\widetilde{F}(-v\sin\theta + w\cos\theta,t) - \widetilde{F}(w,t)\widetilde{F}(w,t)]\,.$$

The argument in [11] formally works also in our case but it cannot be applied directly since it is based on the power series expansion of $e^{\tilde{\lambda}_n K_N t}$ which does not extend to a convergent power series expansion of $e^{\tilde{\lambda}_n\mathscr{K}t}$. Instead we use a Duhamel style formula again and write

$$e^{\tilde{\lambda}_n K_N t}\phi_k = e^{\tilde{\lambda}_n K_k t}\phi_k + (N-k)\tilde{\lambda}\int_0^t e^{\tilde{\lambda}K_N(t-s)}G_k e^{\tilde{\lambda}_n K_k s}\phi_k ds \tag{26}$$

where we wrote

$$K_N = K_k + \widetilde{K}_{N-k} + (N-k)G_k$$

with

$$\widetilde{K}_{N-k} = \sum_{k+1\leq i<j\leq N}(R_{i,j} - \mathrm{Id})$$

$$G_k = \frac{1}{N-k}\sum_{i=1}^{k}\sum_{j=k+1}^{N}(R_{i,j} - \mathrm{Id})$$

Since we are interested in integrating (26) against a symmetric function f_N we can write

$$G_k[\phi_k](\underline{v}_{k+1}) = \sum_{i=1}^{k}\int\frac{d\theta}{2\pi}[\phi(v_1,\ldots,v_{i-1},v_i\cos\theta + v_{k+1}\sin\theta, v_{i+1},\ldots,v_k) - \phi(\underline{v}_k)]$$

To iterate we need to apply (26) to the factor $e^{\tilde{\lambda}K_N(t-s)}$ in the integral in (26) itself. Since $G_k e^{\tilde{\lambda}_n K_k s}\phi_k$ is a function of $k+1$ variable we now write

$$K_N = K_{k+1} + \widetilde{K}_{N-k-1} + (N-k-1)G_{k+1}\,.$$

Iterating this procedure we get

$$
\begin{aligned}
e^{\tilde\lambda_n K_N t}\phi_k =& e^{\tilde\lambda_n K_k t}\phi_k +\\
&\sum_{p=1}^{N-k}\frac{\tilde\lambda_n^p (N-k)!}{(N-k-p)!}\int_{0<t_1<\cdots<t_p<t} e^{\tilde\lambda_n K_{k+p}(t-t_p)}G_{k+p-1}e^{\tilde\lambda_n K_{k+p-1}(t_p-t_{p-1})}\ldots\\
&\qquad e^{(t_2-t_1)\tilde\lambda K_{k+1}}G_k e^{\tilde\lambda_n K_k t_1}\phi_k\, dt_n\cdots dt_1
\end{aligned}
$$

so that

$$
\begin{aligned}
(e^{\tilde\lambda_n \mathscr{K} t}\mathbf{f}_n,\phi_k)_{k,n} =& (\mathbf{f}_n, e^{\tilde\lambda_n K_k t}\phi_k)_{k,n} +\\
&\sum_{p=1}^{\infty}\lambda^p\int_{0<t_1<\cdots<t_p<t}\Big(\mathbf{f}_n, e^{\tilde\lambda_n K_{k+p}(t-t_p)}G_{k+p-1}e^{\tilde\lambda_n K_{k+p-1}(t_p-t_{p-1})}\ldots\\
&\qquad e^{(t_2-t_1)\tilde\lambda K_{k+1}}G_k e^{\tilde\lambda_n K_k t}\phi_k\Big)_{k+p,n} dt_n\cdots dt_1
\end{aligned}
$$

Thus, since $\|e^{\tilde\lambda_n K_N}\|_\infty \le 1$, we get

$$
\left|(e^{\tilde\lambda_n \mathscr{K} t}\mathbf{f}_n,\phi_k)_{k,n}\right| \le \|\mathbf{f}\|_1\|\phi_k\|_\infty + \sum_{p=1}^{\infty}\frac{\lambda^p t^p}{p!}\prod_{i=k}^{k+p-1}\|G_i\|_\infty\|\mathbf{f}\|_1\|\phi_k\|_\infty
$$

where, due to the fact that $\|G_i\|_\infty \le 2i$, the series on the right hand side converges for $t<1/2$, see [11].

We can now take the limit $n\to\infty$ and, calling $G_k^{*p}=\prod_{i=0}^p G_{k+i}$ and using Dominated Convergence, we get

$$
\begin{aligned}
\lim_{n\to\infty}(e^{\tilde\lambda_n \mathscr{K} t}\mathbf{f}_n,\phi_k)_{k,n} =& \lim_{n\to\infty}\sum_{p=0}^{\infty}\frac{\lambda^p t^p}{p!}\left(\mathbf{f}_n, G_k^{*p}\phi_k\right)_{k+p,n} =\\
&\sum_{p=0}^{\infty}\frac{\lambda^p t^p}{p!}\int_{\mathbb{R}^{k+p}}\prod_{j=1}^{p+k}F(v_j)G_k^{*p}\phi_k
\end{aligned}
$$

where we used that $\tilde\lambda_n\to 0$ as $n\to\infty$ and that $\mathbf{f}_n$ form a chaotic sequence.

We can now proceed exactly as in [11] to obtain (25), observing that if $\mathbf{f}_n$ form a chaotic sequence and ϕ is in $C(\mathbb{R}^{k_1})$ while ψ is in $C(\mathbb{R}^{k_2})$ with $k_1+k_2=k$ then

$$
\lim_{n\to\infty}(\mathbf{f}_n,\phi\otimes\psi)_{k,n} = \int F^{\otimes k}(\underline{v}_k)\phi\otimes\psi(\underline{v}_k)d\underline{v}_k = \lim_{n\to\infty}(\mathbf{f}_n,\phi)_{k_1,n}\lim_{n\to\infty}(\mathbf{f}_n,\psi)_{k_2,n}
$$

and that G_k acts as a derivation in the sense of [11].

We now turn to the evolution generated by the thermostat $\mathscr{T}$ and observe that

$$
\left(e^{t\rho(\mathscr{O}-\mathscr{N})}\mathbf{f}_n\right)_N(\underline{v}_N) = e^{-\rho t}\sum_{M>N}\binom{N}{M}(1-e^{\rho t})^{M-N}\int f_{n,M}(\underline{v}_M)dv_{N+1}\cdots dv_M
$$

from which it follows that

$$
(e^{t\rho(\mathscr{O}-\mathscr{N})t}\mathbf{f}_n,\phi)_{k,n} = e^{-t\rho k}(\mathbf{f}_n,\phi)_{k,n}
$$

while

$$\mu_n(\mathscr{I}\mathbf{f}_n - \mathbf{f}_n, \phi)_{k,n} = (\mathbf{f}_n, I\phi)_{k-1,n}$$

where

$$I[\phi](\underline{v}_{k-1}) := \rho k \int_{\mathbb{R}} \phi(\underline{v}_{k-1}, w) e^{-\pi w^2} dw\,.$$

Combining we get

$$\begin{aligned}(e^{\mathscr{T}_n t}\mathbf{f}_n, \phi^{\otimes k})_{k,n} =& (e^{\rho(\mathscr{O}-\mathscr{N})t}\mathbf{f}_n, \phi^{\otimes k})_{k,n} + \\ & \sum_{p=1}^{\infty} \int_{0\le t_1 \le \ldots \le t_n \le t} \Big(e^{\rho(\mathscr{O}-\mathscr{N})(t-t_n)} \mu_n(\mathscr{I} - \mathrm{Id}) e^{\rho(\mathscr{O}-\mathscr{N})(t_n - t_{n-1})} \ldots \\ & e^{\rho(\mathscr{O}-\mathscr{N})(t_2-t_1)} \mu_n(\mathscr{I} - \mathrm{Id}) e^{\rho(\mathscr{O}-\mathscr{N})t_1} \mathbf{f}_n, \phi^{\otimes k} \Big)_{k,n} = \\ & \sum_{p=0}^{k} \binom{k}{p} \Big(\mathbf{f}_n, e^{\rho(k-p)t} \phi^{k-p} \Big)_{k-p,n} \left((1 - e^{-\rho t}) \int \phi(v)\gamma(v) dv \right)^p\end{aligned}$$

from which it follows that

$$\lim_{n\to\infty} (e^{\mathscr{T}_n t}\mathbf{f}_n, \phi^{\otimes k})_{k,n} = \left(\int_{\mathbb{R}} \widehat{F}(v,t)\phi(v) dv \right)^k$$

where $\widehat{F}(v,t)$ satisfies

$$\frac{d}{dt}\widehat{F}(v,t) = -\rho(\widehat{F}(v,t) - \gamma(v))\,. \qquad (27)$$

Eq. (27) can be thought as a generalization of (8) for a chaotic state $\mathbf{f}$. Indeed, if $\mathbf{f}$ is of the form (7) then $\widehat{F}(v,t) = \rho\eta(t)g(t,v)/\mu$.

Finally using that

$$e^{t(\mathscr{T}_n + \tilde{\lambda}\mathscr{K})} = \lim_{m\to\infty} \left(e^{\frac{t}{m}\mathscr{T}_n} e^{\frac{t}{m}\tilde{\lambda}\mathscr{K}} \right)^m$$

to combine (25) and (27), we obtain the Boltzmann-Kac equation (12). □

4 Conclusions

The central aim of this work is the extension of the analysis in [2], in which a thermostat idealizes the interaction with a large reservoir of particles kept at constant temperature and chemical potential. While in [2] the reservoir and the system could not exchange particles, here the main interaction is the continuous exchange of particles between the two.

However, it in is in these same works which we hoped to extend that we also find points of possible extension to our current work. In the case of the Maxwellian thermostat described in [2], the authors show that convergence to equilibrium occurs in the GTW metric d_2 as well (see [4,2]). In the present situation though, it is not clear how to define an analogue of the GTW metric since the components f_N of a state $\mathbf{f}$ are not, in general, a probability distribution on $\mathbb{R}^n$.

Furthermore, in [1] the authors show that, in a strong and uniform sense, the evolution of the Kac system with a Maxwellian thermostat can be thought of as an idealization of the interaction with a large heat reservoir. We think it possible to replicate such an analysis in the present context and hope to come back to this issue in a forthcoming paper.

We based our proof of propagation of chaos on the work in [11]; therefore, as in [11], it is not quantitative nor uniform in time. Recently, a quantitative and uniform in time result was obtained for the Kac system with a Maxwellian thermostat [5]. It is unclear to us whether the methods in their work extend to the present model.

Finally, that the rates ρ and μ are independent of the number of particles in the system is clearly unrealistic, allowing the possibility of an unbounded number of particles in the system. However, in the steady state (and in a chaotic state) such a probability is extremely small, and so we do not consider this a serious problem. In any case, it would be interesting to investigate what happens if one assumes a maximum number of particles allowed inside the system.

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