No books or notes allowed. No laptop, graphic calculator or wireless devices allowed. Write clearly. Show your work and justify your answers

Name: ____________________

Question:	1	2	3	4	5	Total
Points:	20	11	12	42	15	100
Score:						

1. Evaluate each of the following statements as true or false. Justify your answer by either giving a brief explanation or providing a counterexample as appropriate.

   (a) (10 points) There exists a differentiable function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) < 2$ for all x.

   Solution: False. If $f(0) = -1$ and $f(2) = 4$ by the Mean Value Theorem there exists a c in $[0, 2]$ such that

   $$f'(c) = \frac{f(2) - f(0)}{2} = \frac{5}{2} > 2. \tag{1}$$

   This contradict the assumption that $f'(x) < 2$ for all x.

   (b) (10 points) If $f'(0) = 0$ and $f''(0) = 0$, then f has neither a local maximum nor a local minimum at $x = 0$.

   Solution:
   False. Let $f(x) = x^4$. Then $f'(0) = 0$ and $f''(0) = 0$ but $x = 0$ is a local minimum.

2. (11 points) Use differentials to approximate the following number:

$$\sqrt{3.98} \tag{2}$$

Show your work. No credit will be given for writing only the result.

Solution: Let $f(x) = \sqrt{4-x}$. We have $f(0) = 2$ and $f'(0) = -1/4$. Thus

$$\sqrt{4.98} = f(0.2) \simeq f(0) + 0.02 \cdot f'(0) = 2 - \frac{0.02}{4} = 1.995. \tag{3}$$

3. (12 points) Compute the following limit

$$\lim_{n\to\infty} \frac{2n^2+1}{4n^2+5}$$

Solution:

$$\lim_{n\to\infty} \frac{2n^2+1}{4n^2+5} = \lim_{n\to\infty} \frac{2n^2}{4n^2} \lim_{n\to\infty} \frac{1+\frac{1}{2n^2}}{1+\frac{5}{4n^2}} = \frac{1}{2}$$

4. Let f be the function:

$$f(x) = \frac{3x^2 + 2}{x^2 - 9} \tag{4}$$

(a) (12 points) Find the domain, intercepts and asymptotes of f.

Solution: Domain: $D(f) = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Intercepts: $f(0) = -\frac{2}{9}$, $f(x)$ is never equal to 0.
We have:

$$\begin{aligned} \lim_{x \to -3^-} f(x) &= +\infty & \lim_{x \to -3^+} f(x) &= -\infty \\ \lim_{x \to 3^-} f(x) &= -\infty & \lim_{x \to 3^+} f(x) &= \infty \\ \lim_{x \to -\infty} f(x) &= 3 & \lim_{x \to +\infty} f(x) &= 3 \end{aligned} \tag{5}$$

So that $x = -3$ and $x = 3$ are vertical asymptotes while $y = 3$ is an horizontal asymptote.

(b) (12 points) Find critical values, local maxima and minima of f and where f is increasing and decreasing.

Solution:

$$f'(x) = \frac{6x(x^2 - 9) - 2x(3x^2 + 2)}{(x^2 - 9)^2} = -\frac{58x}{(x^2 - 9)^2} \tag{6}$$

Since $(x^2 - 9)^2 > 0$ for every x in $D(f)$, we have $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$.
Thus $x = 0$, $x = -3$ and $x = 3$ are the critical values, $x = 0$ is a local maximum and there is no local minimum.
Finally f is increasing on $(-\infty, -3)$ and on $(-3, 0]$ while f is decreasing on $[0, 3)$ and on $(3, \infty)$.

(c) (8 points) Find where f is concave up and concave down and its inflection points.

Solution:

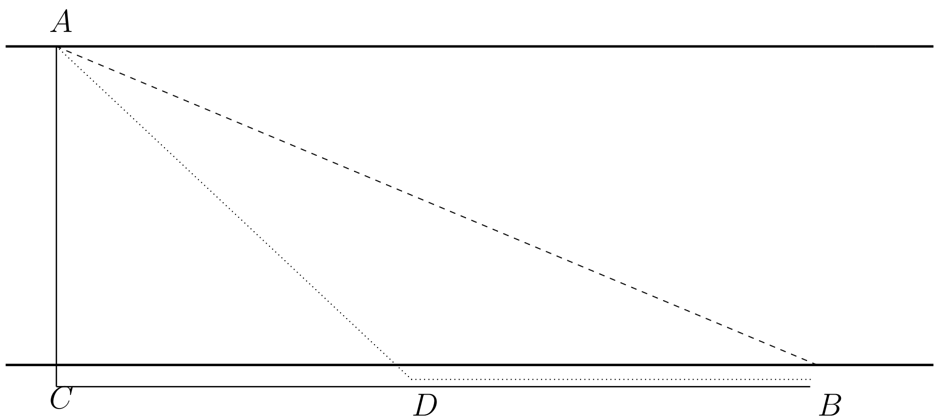
$$f''(x) = \frac{-58(x^2-9)^2 + 232x^2(x^2-9)}{(x^2-9)^4} = \frac{174(x^2+3)}{(x^2-9)^3} \tag{7}$$

Since $174(x^2+3) > 0$ for every x, we have $f''(x) < 0$ iff $x^2-9 < 0$ or $-3 < x < 3$ and $f''(x) > 0$ for $x < -3$ or $x > 3$. Thus there is no inflexion point and f is concave up on $(-\infty, -3)$ and $(3, \infty)$ while f is concave down on $(-3, 3)$.

(d) (10 points) Sketch the graph of f.

Solution:

5. (15 points) A man launches his boat from point A on a bank of a straight river, 6 miles wide, and wants to reach point B, 20 miles downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 4 miles for hour and run 5 miles for hour, where should he land to reach B as soon as possible? (Assume the speed of the water is negligible compared with the speed at which the man rows.)



Solution: Let x be the distance between C and D in miles. Then the distance the man has to row in the water is $d_w = \sqrt{9+x^2}$ while the distance he has to run on land is $d_l = 10 - x$. Considering the two velocities, we have that the total time in hours he needs to go form A to B passing through D is

$$T(x) = \frac{d_l}{5} + \frac{d_w}{4} = \frac{10-x}{5} + \frac{\sqrt{9+x^2}}{4} \tag{8}$$

We thus have

$$T'(x) = -\frac{1}{5} + \frac{x}{4\sqrt{9+x^2}} \tag{9}$$

where $0 \le x \le 10$. It is easy to see that $T(x) = 0$ only for $x = 4$. Thus $x = 4$ is the only critical point in the interesting domain, with $T(4) = \frac{49}{20}$.

At the boundary we have $T(0) = \frac{11}{4} > \frac{49}{20}$ and $T(10) = \frac{\sqrt{109}}{4} > \frac{49}{20}$. So we can conclude, since $x = 4$ is the only critical point, that the optimal position for D is at a distance of 4 miles from C.