MATH 3235, Test I **Houdré**
Tuesday February 27th, 2018 **Name:** Solutions

Answer all questions; show all work; closed books, no calculators. THE HONOR CODE APPLIES TO THIS CLASS

Problem	Points	Score
1	50	50
2	50	50
3	50	50
4	50	50
Total	200	200

1. (50pts) Let A, B and C be three independent events which occur with respective probability $\mathbb{P}(A) = 1/4$, $\mathbb{P}(B) = 1/2$ and $\mathbb{P}(C) = 1/3$.

   (a) (25pts) Find the probability that at least one of the events occurs.

$$\mathbb{P}(\text{at least one}) = 1 - \mathbb{P}(\text{none})$$
$$= 1 - \mathbb{P}(A^c \cap B^c \cap C^c)$$
$$\perp\!\!\!\perp \quad = 1 - \mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c)$$
$$= 1 - 3/4 \cdot 1/2 \cdot 2/3 = 1 - 1/4 = 3/4$$

(b) (25pts) Find the probability that exactly two of the events occur.

$\mathbb{P}(\text{exactly } 2) = \mathbb{P}((A^c \cap B \cap C) \cup (A \cap B^c \cap C)$
$\cup (A \cap B \cap C^c))$

pairwise disjoint

$= \mathbb{P}(A^c \cap B \cap C) + \mathbb{P}(A \cap B^c \cap C)$
$+ \mathbb{P}(A \cap B \cap C^c)$

$\perp\!\!\!\perp$

$= \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B^c)\mathbb{P}(C)$
$+ \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c)$

$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{2}{3}$

$= \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{3+1+2}{24} = \frac{6}{24} = \frac{1}{4}$

2. (50pts) A medical doctor wishes to detect a particular chemical substance in a blood sample. A test indicates the presence of this chemical substance, when it is there, 90% of the time; however this test also produces false positive and indicates the presence of this chemical substance, when it is not there, 5% of the time. It is known that 80% of the samples do not contain this substance (and 20% do).

   (a) (25pts) What is the probability that for a randomly selected sample, the test is positive, i.e., detects the substance?

B = the test is positive.

A : the sample contains the substance

$B = (A \cap B) \cup (A^c \cap B)$

$\mathbb{P}(B) \underset{\text{pairwise disjoint}}{=} \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$

$= \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)$

$= (0.9)(0.2) + (0.05)(0.8)$

$= 0.18 + 0.04 = 0.22$

(b) (25pts) What is the probability that a randomly selected sample does actually contain the substance, given that the test is positive?

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$= \frac{(0.9)(0.2)}{0.22} = \frac{18}{22} = \frac{9}{11}$$

the first is / second is

3. (50pts) I throw two fair dice, one four-sided and the other three sided, and record their respective score S_1 and S_2. Let X be the sum of the scores, i.e., $X = S_1 + S_2$ and let Y be the difference of the scores, i.e., $Y = S_1 - S_2$. Find the joint probability mass function of X and Y. Are X and Y independent?

S_1 takes the values $1,2,3,4$ each with probability $1/4$.
S_2 " $1,2,3$ " " $1/3$.

So $X = S_1 + S_2$ takes the values $2,3,4,5,6,7$
while $Y = S_1 - S_2$ takes the values $-2,-1,0,1,2,3$

X \ Y	-2	-1	0	1	2	3
2	0	0	1/12	0	0	0
3	0	1/12	0	1/12	0	0
4	1/12	0	1/12	0	1/12	0
5	0	1/12	0	1/12	0	1/12
6	0	0	1/12	0	1/12	0
7	0	0	0	1/12	0	0

or:

$$p_{(X,Y)}(x,y) = \mathbb{P}(X = x, Y = y) = \mathbb{P}(S_1 + S_2 = x,\ S_1 - S_2 = y)$$

$$= \mathbb{P}\left(S_1 = \frac{x+y}{2},\ S_2 = \frac{x-y}{2}\right) \underset{\perp\!\!\!\perp}{=} \mathbb{P}\left(S_1 = \frac{x+y}{2}\right)\mathbb{P}\left(S_2 = \frac{x-y}{2}\right)$$

$$= \begin{cases} \frac{1}{4}\cdot\frac{1}{3} & \text{but only for } \frac{x+y}{2} = 1,2,3,4 \text{ and } \frac{x-y}{2} = 1,2,3. \\ 0 & \text{elsewhere.} \end{cases}$$

Not $\perp\!\!\!\perp$ since $\mathbb{P}(X=2, Y=-2) = 0 \neq \mathbb{P}(X=2)\mathbb{P}(Y=-2)$
$= \frac{1}{12}\cdot\frac{1}{12} = \frac{1}{144}$

4. (50pts) The sea floor of a particular Tahitian lagoon contains N oysters, where N is a Poisson random variable with parameter 1100, and each oyster contains a black pearl, independently of the other oysters, with fixed probability 1/1000. Oysters are harvested from the bottom of the lagoon, opened, and let X be the number containing a black pearl.

0

(a) (20pts) Find $p_{X|N=n}$, the conditional pmf of X given $\{N = n\}$, $n = 0, 1, 2, \ldots$

If $n=0$; then $X=0$, and $\mathbb{P}(X=0|N=0) = 1$. If $n=1$, then $X = 0$ or 1, $\mathbb{P}(X=0|N=1) = 1-10^{-2}$, $\mathbb{P}(X=1|N=1) = 10^{-2}$

If $n=2$, then $X = 0, 1$ or 2.

$p_{X|N=2}(0) = (1-10^{-2})^2$; $p_{X|N=2}(1) = 2 \cdot 10^{-2}(1-10^{-2})$

$p_{X|N=2}(2) = (10^{-2})^2$.

More generally, for n arbitrary ≥ 3., X takes the values $0, 1, 2, \ldots, n$ and

$$p_{X|N=n}(x) = \binom{n}{x}(10^{-2})^x(1-10^{-2})^{n-x}$$

(b) (20pts) Find $\mathbb{E}(X|N)$.

So for $n \geq 1$, $X|N=n$ has a binomial pmf with parameters n and $p = 10^{-}$.

(b)
$$\mathbb{E}(X|N=n) = \sum_{x=0}^{n} x \binom{n}{x}(10^{-2})^x(1-10^{-2})^{n-x}$$

$$= n \cdot 10^{-2},$$

and $= 0$ for $n = 0$.

So

$Y = \mathbb{E}(X|N) = 10^{-2} N$, is a multiple of a Poisson r.v., i.e., Y takes the values $10^{-2}k$, $k = 0, 1, 2, \ldots$ with $\mathbb{P}(Y = 10^{-2}k) = e^{-10^3} \frac{(10^3)^k}{k!}$.

(c) (10pts) Find $\mathbb{E}X$, the expectation of X.

Now

$$\mathbb{E}X = \mathbb{E}(10^{-2}N) = 10^{-2}\mathbb{E}N = 10^{-2} \cdot 10^3$$

(if you remember that $\mathbb{E}N = \lambda$) $= 10$.

Could also do it using the law of "total expectation"

$$\mathbb{E}X = \sum_{n=0}^{\infty} \mathbb{E}(X|N=n)\,\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} n\, 10^{-2} \cdot e^{-10^3} \frac{(10^3)^n}{n!} = 10^{-2} e^{-10^3} \cdot \sum_{n=0}^{\infty} n \frac{(10^3)^n}{n!}$$

$$= 10^{-2} e^{-10^3}\, 10^3\, e^{10^3}$$

$$= 10$$