Exercises

5.13 Clearly $Y = 0$ if $X \leq 0$ while
$Y = X$ if $X \geq 0$.
Thus calling $F_Y(y) = \mathbb{P}(Y \leq y)$ we have

$$F_Y(y) = 0 \quad \text{if } y < 0$$
$$F_Y(y) = F_X(y) \quad \text{if } y \geq 0$$

5.18
Clearly
$$0 \leq F(x) \leq 1$$
since $0 \leq a \leq 1$. Moreover if $x \leq y$ we have
$$F(x) = \alpha F_1(x) + (1-\alpha) F_2(x) \leq \alpha F_1(y) + (1-\alpha) F_2(y) = F(y).$$

Also
$$\lim_{x \to -\infty} F(x) = \alpha \lim_{x \to \infty} F_1(x) + (1-\alpha) \lim_{x \to \infty} F_2(x)$$
and similarly for $x \to +\infty$.

Finally
$$\lim_{y \to x^+} F(y) = \alpha \lim_{y \to x^+} F_1(y) + (1-\alpha) \lim_{y \to x^+} F_2(y) =$$
$$\alpha F_1(x) + (1-\alpha) F_2(x) = F(x)$$

Thus F is a distribution function.

5.30 We have

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx$$

Thus $F_X(x) = 0$ if $x \le 0$.

If $0 \le x \le 1$ we have

$$F_X(x) = \int_0^x 2y\,dy = y^2 \Big|_0^x = x^2$$

Finally

$$F_X(x) = 1 \quad \text{for } x \ge 1$$

5.32 Clearly $F(x)$ is differentiable everywhere but at most 0. We have

$$\frac{1}{2(1+x^2)}\Bigg|_{x=0} = \frac{1}{2} = \frac{1+2x^2}{2(1+x^2)}$$

Thus F is continuous.

Moreover

$$\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \frac{1}{2(1+x^2)} = 0$$

while

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \frac{1+2x^2}{2(1+x^2)} = 1$$

Finally

$$\frac{d}{dx}F(x) = \begin{cases} -\dfrac{x}{(1+x^2)^2} & x<0 \\ \dfrac{x}{(1+x^2)^2} & x \end{cases}$$

Since $\frac{d}{dx}F(x) \geq 0$ for every x ~~we~~ and it is differentiable everywhere ~~but~~ ~~and~~ we have that X is a continuous r.v. and

$$f(x) = \begin{cases} -\dfrac{x}{(1+x^2)^2} & x<0 \\ \dfrac{x}{(1+x^2)^2} & x \geq 0 \end{cases}$$

5.54

a) ~~$f_A(x) =$~~ Since $X = \frac{A-5}{2}$

we have

$$f_A(x) = \begin{cases} \frac{\lambda}{2} e^{-\frac{\lambda}{2}(x-5)} & x \geq 5 \\ 0 & \text{otherwise} \end{cases}$$

b) $X = \ln B$ so that

$$f_B(x) = \begin{cases} x^{-\lambda-1} & x \geq 1 \\ 0 & x \leq 1 \end{cases}$$

c) Since

$$X = \frac{1}{C} - 1$$

we get

$$f_C(x) = \begin{cases} \frac{1}{x^2} e^{-\lambda\left(\frac{1}{x} - 1\right)} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

d) similarly

$$X = \frac{1}{\sqrt{D}} - 1$$

and

$$f_D(x) = \begin{cases} \frac{1}{2} \frac{1}{D^{3/2}} e^{-\lambda\left(\frac{1}{\sqrt{x}} - 1\right)} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problems

7. We Know That

$$\mathbb{E}(X) = \int_0^\infty x\, f_X(x)\,dx$$

where we used That $f(x)=0$ for $x<0$ since f Takes only positive values. Since

$$f_X(x) = \frac{d}{dx}\left(F_X(x) - 1\right)$$

we can integrate by part and get

$$\int_0^\infty f_X(x)\,dx = x\left(F_X(x)-1\right)\Big|_0^\infty - \int_0^\infty \left(F_X(x) - 1\right) dx$$

Observe That if

$$-\int_0^\infty \left(F_X(x) - 1\right) dx < +\infty$$

Then

$$\lim_{x\to\infty} x\left(F_X(x) - 1\right) = 0 .$$

11. If The angle is θ we have ThaT The distance is $x = \tan\theta$.

Thus

$$\theta = \arctan x$$

Moreover

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

so ThaT

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

for $-\infty < x < +\infty$.

Finally we get

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+y^2}\, dy =$$

$$\frac{1}{\pi} \arctan x + \frac{1}{2}$$

6.14 Observe that calling

$$A_1 = \{(x,y) \mid a < x \le b, \; c < y \le d\}$$

$$A_2 = \{(x,y) \mid x \le b, \; y \le c\}$$

$$A_3 = \{(x,y) \mid x \le a, \; y \le d\}$$

and finally

$$B = \{(x,y) \mid x \le b, \; y \le d\}$$

we have

$$B = A_1 \cup A_2 \cup A_3$$

since $A_1 \cap A_2 = 0$ and $A_1 \cap A_3 = 0$ we get

$$\mathbb{P}(B) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_2 \cap A_3)$$

The Thesis follows observing that

$$A_2 \cap A_3 = \{(x,y) \mid x \le a, \; y \le c\}$$

6.26

$$\mathbb{P}(X+Y \leq 1) = \int_0^1 dx \int_0^{1-x} dy\, e^{-x-y} =$$

$$\int_0^1 dx\, e^{-x} (\cancel{1} - e^{-y}) \Big|_0^{1-x} =$$

$$\int_0^1 dx \left(e^{-x} - e^{-1}\right) = 1 - 2e^{-1}$$

$\mathbb{P}(X \leq Y) = \mathbb{P}(Y \leq X)$ by symmetry

but

$$\mathbb{P}(X \leq Y) + \mathbb{P}(Y \leq X) = 1$$

so that $\mathbb{P}(X \leq Y) = \frac{1}{2}$