6.14 Observe that calling

$$A_1 = \{(x,y) \mid a < x \leq b ,\ c < y \leq d\}$$

$$A_2 = \{(x,y) \mid x \leq b ,\ y \leq c\}$$

$$A_3 = \{(x,y) \mid x \leq a ,\ y \leq d\}$$

and finally

$$B = \{(x,y) \mid x \leq b ,\ y \leq d\}$$

we have

$$B = A_1 \cup A_2 \cup A_3$$

since $A_1 \cap A_2 = 0$ and $A_1 \cap A_3 = 0$ we get

$$\mathbb{P}(B) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_2 \cap A_3)$$

The Thesis follows observing that

$$A_2 \cap A_3 = \{(x,y) \mid x \leq a ,\ y \leq c\}$$

6.26

$$\mathbb{P}(X+Y\le 1) = \int_0^1 dx \int_0^{1-x} dy\, e^{-x-y} =$$

$$\int_0^1 dx\, e^{-x}(\cancel{1}-e^{-y})\Big|_0^{1-x} =$$

$$\int_0^1 dx\,(e^{-x}-e^{-1}) = 1-2e^{-1}$$

$\mathbb{P}(X\le Y) = \mathbb{P}(Y\le X)$ by symmetry

but

$$\mathbb{P}(X\le Y)+\mathbb{P}(Y\le X)=1$$

so that $\mathbb{P}(X\le Y) = \frac{1}{2}$

6.36 Let

$$h(x) = \frac{1}{2}x \qquad 0\le x\le 1$$

we have

$$f(x,y,z) = h(x)\,h(y)\,h(z)$$

so that X, Y, Z are independent.

As for the previous question the

density is symmetric in x and y so that

$$\mathbb{P}(X > Y) = \frac{1}{2}$$

and similarly

$$\mathbb{P}(Y > Z) = \frac{1}{2}$$

6.45 Calling $Z = X + Y$ we have

$$f_Z(z) = \int_{\mathbb{R}} f(t, z-t)\, dt =$$

$$= \int_0^z \frac{1}{2} z e^{-z}\, dt = \frac{1}{2} z^2 e^{-z}$$

6.55 From the definition we get

$$Y = V$$
$$X = 2U + V$$

so that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

and

$$f_{U,V}(u,v) = \frac{1}{2} e^{-u-v}$$

for $v > 0$ and ~~any~~ $u > -v/2$.

Thus

$$f_U(u) = \frac{1}{2}\int_{-2u}^{\infty} e^{-\frac{1}{2}u - v}\, dv \qquad \text{if } u<0$$

while

$$f_U(u) = \frac{1}{2}\int_{0}^{\infty} e^{-u-v}\, dv \qquad \text{if } u>0$$

~~Thus~~ So we get

$$f_U(u) = \frac{1}{2}\begin{cases} e^{-u} & u>0 \\ e^{u} & u<0 \end{cases}$$

That is

$$f_U(u) = \frac{1}{2}e^{-|u|}$$

6.61 We have

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Calling $U = X$ and $V = X+Y$ we get $X = U$ and $Y = V-U$. Clearly

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 1$$

so That

$$f_{U,V}(u,v) = \begin{cases} \lambda^2 e^{-\lambda v} & 0 \le u \le v \\ 0 & \text{otherwise} \end{cases}$$

From This we get:

$$f_V(v) = \int_0^v \lambda^2 e^{-\lambda v} \, du = \lambda^2 v e^{-\lambda v}$$

for $v > 0$ and

$$f_{U|V}(u,v) = \frac{\lambda^2 e^{-\lambda v}}{\lambda^2 v e^{-\lambda v}} = \frac{1}{v} \qquad 0 \le u \le v.$$

6.70 We have

$$\mathbb{E}\left(\sqrt{X^2+Y^2}\right) = \frac{1}{\pi} \int_{x^2+y^2 \le 1} \sqrt{x^2+y^2} \, dx \, dy =$$

$$\frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^1 dr \, r \cdot r. =$$

$$2 \int_0^1 dr \, r^2 = \frac{2}{3}$$

Similarly

$$\mathbb{E}(X^2+Y^2) = 2 \int_0^1 r^3 \, dr = \frac{1}{2}$$

6.80 Observe That

$$aX + bY = a\sigma_1 \frac{X-\mu_1}{\sigma_1} + b\sigma_2 \frac{Y-\mu_2}{\sigma_2} + a\mu_1 + b\mu_2$$

Calling $U = \frac{X-\mu_1}{\sigma_1}$ and $V = \frac{Y-\mu_2}{\sigma_2}$ we

get

$$aX + bY = a\sigma_1 U + b\sigma_2 V + a\mu_1 + b\mu_2$$

IT is easy To see That (see Ex 6.79

(U,V) are standard bivariate normal with parameter ρ

Since $Z + c$ is normal when Z is normal it is enough To show That

$$\tilde{a}U + \tilde{b}V$$

is normal with $\tilde{a} = a\sigma_1$, $\tilde{b} = b\sigma_2$.

Call now

$$Z = \frac{1}{2}(U + V)$$

$$W = \frac{1}{2}(U - V)$$

so That

$$U = Z + W$$
$$V = Z - W$$

Changing variable we have That

$$f_{Z,W}(z,w) = \frac{1}{\pi} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{z^2}{1-\rho} - \frac{w}{1+\rho}\right)$$

so That Z and W are independent normal r.v.

Observe That

$$\tilde{a}\, U + \tilde{b}\, V = (\tilde{a} + \tilde{b}) Z + (\tilde{a} - \tilde{b}) W$$

The Thesis follows from the fact That The sum of independent normal r.v. is normal.

6. Observe That

$$\mathbb{P}(\max\{X_1, X_2, \ldots X_n\} \leq t) =$$

$$\mathbb{P}(X_1 \leq t \ \& \ X_2 \leq t \ \& \ \ldots \ \& \ X_n \leq t) =$$

$$\mathbb{P}(X_1 \leq t)\,\mathbb{P}(X_2 \leq t) \ldots \mathbb{P}(X_n \leq t) =$$

$$\mathbb{P}(X_i \leq t)^n = F(t)^n$$

So That

$$\cancel{\mathbb{P}} \quad f_V(v) = n F^{n-1}(v)\, f(v)$$

Similarly

$$\mathbb{P}(\min\{X_1 \ldots X_n\} > t\}) =$$

$$\mathbb{P}(X_1 > t)^n$$

so that

$$F_U(u) = 1 - (1 - F(u))^n$$

and

$$f_U(u) = n(1 - F(u))^{n-1} f(u)$$

Finally

$$\mathbb{P}(V \leq t_1 \ \& \ U \leq t_2) =$$

$$\mathbb{P}(V \leq t_1) - \mathbb{P}(V \leq t_1 \ \& \ U > t_2) =$$

but

$$\mathbb{P}(V \le t_1 \;\&\; U > t_2) = \mathbb{P}(t_2 \le X_1 \le t_1)^n \quad t_2 < t_1$$

$$= 0 \quad t_1 \ge t_2$$

so That

$$F(u,v) = F(v)^n - (F(v) - F(u))^n \quad v > u$$

$$= F(v)^n \quad v < u$$

Finally

$$f(u,v) = \partial_u \partial_v F(u,v) =$$

$$= n(n-1) f(u) f(v) [F(v) - F(u)]^{n-2}$$

20. Since the region $\{(x,y) \mid 0<y<x<1\}$ has area $\frac{1}{2}$ we have

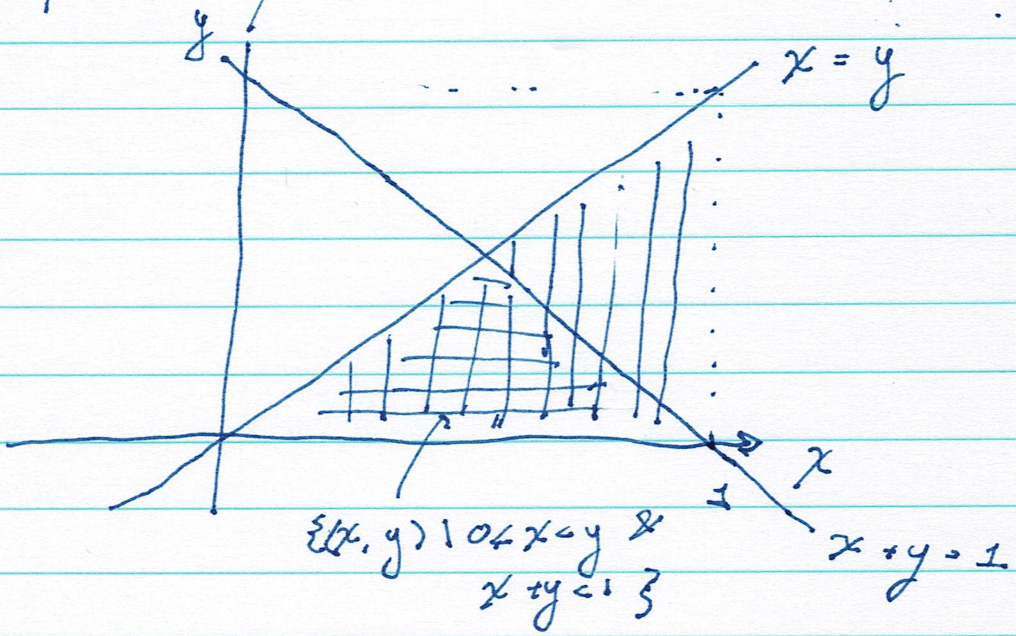
$$f_{X,Y}(x,y) = \begin{cases} \cancel{4}\,2 & \text{if } 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\mathbb{P}(X+Y<1) =$$

$$= \frac{2}{\cancel{4}} \int_0^{1/2} dy \int_y^{1-y} dx = \frac{2}{\cancel{4}} \int_0^{1/2} dy\,(1-2y) = \cancel{4}\,\frac{1}{2}$$

Graphically



The region

$$\{(x,y) \mid 0 < x < y \;\; \& \;\; x + y < 1\}$$

is half the region

$$\{(x,y) \mid 0 < x < y\}$$

7.10

If X is uniform in (a,b) Then

$$\mathbb{E}(X^K) = \frac{1}{b-a}\int_a^b x^K dx =$$

$$= \frac{1}{b-a}\frac{x^{K+1}}{(K+1)}\Big|_a^b = \frac{b^{K+1} - a^{K+1}}{(b-a)(K+1)}$$

7.36

We have

$$\mathrm{cov}(S_m, S_n) =$$

$$\mathrm{cov}(S_m, S_m + \sum_{i=m+1}^{n} X_i) =$$

$$\mathrm{var}(S_m) + \sum_{j=1}^{m}\sum_{i=m+1}^{n} \mathrm{cov}(X_j, X_i) =$$

$$\mathrm{var}(S_m) = m\sigma^2$$