7.4 n 6

Since the prior distribution is a Gamma distribution with mean μ_0 we know that $\frac{\alpha}{\beta} = \mu_0$. The posterior distribution will be a Gamma with par. $\alpha' = \alpha + n\overline{X}_n$ and $\beta' = \beta + n$ so that the mean of the posterior distribution is

$$\frac{\alpha'}{\beta'} = \frac{\alpha + n\overline{X}_n}{\beta + n} = \frac{\alpha}{\beta + n} + \frac{n}{\beta + n}\overline{X}_n$$

Calling $\gamma_n = \frac{n}{\beta + n}$ and observing that

$$\frac{\alpha}{\beta + n} = \frac{\alpha}{\beta}\,\frac{\beta}{\beta + n} = \mu_0(1 - \gamma_n)$$

we get $\frac{\alpha'}{\beta'} = \gamma_n \overline{X}_n + (1 - \gamma_n)\mu_0$

Clearly $\gamma_n \to 1$ as $n \to \infty$.

12

Observe That

2θ is $B(2,1)$

$4\theta^3$ is $B(4,1)$

a) Let $\underset{\sim}{x}$ be The result of The sample. Since $\sum_{i=1}^{1000} x_i = 710$ we get

$\xi_A(\theta \mid \underline{x})$ is $B(712, 291)$

$\xi_B(\theta \mid \underline{x})$ is $B(714, 291)$

b) $\hat{\theta}_A(\underline{x}) = 712/1003$

$\hat{\theta}_B(x) = 714/1005$

c) If n is The number of opinion in favor we have

$\hat{\theta}_A(\underline{x}) = \dfrac{n+2}{1003}$ $\qquad \hat{\theta}_B = \dfrac{n+4}{1005}$

So That we have

$$\left|\hat{\theta}_A - \hat{\theta}_B\right| = \frac{|2n - 2002|}{1003 \cdot 1005} \leq \frac{2002}{1003 \cdot 1005} \leq 0.002$$

n 13

Since the prior distribution is Pareto x_0, α we know that

$$x_0' = \max(x_0, \underline{X})$$

$$\alpha' = \alpha + n$$

Since the expected value of a Pareto distribution is

$$\frac{\alpha}{\alpha - 1} x_0$$

we get that the Bayes estimator is

$$\hat{\theta}(\underline{X}) = \frac{\alpha_0 + n}{\alpha_0 + n - 1} \max(x_0, \underline{X})$$

7.5 n 5

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{\theta^{x_i}}{x_i!} e^{-\theta} \right) =$$

$$= \frac{\theta^{\sum_i x_i}}{\prod_i x_i!} e^{-n\theta}$$

so that

$$\ell(\theta) = \ln L(\theta) = \sum_i x_i \ln\theta - n\theta - \ln \prod_i x_i!$$

We get

$$\ell'(\theta) = \frac{1}{\theta} \sum_i x_i - n$$

If $\sum_i x_i \neq 0$ this gives

$$\hat{\theta}(\underline{x}) = \frac{\sum_i x_i}{n} = \bar{x}$$

If all $x_i = 0$ we get $\hat{\theta} = 0$ that is not a proper Poisson distribution n 9

The likelihood function is

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

so that

$$\ell(\theta) = \ln L(\theta) = n \ln \theta + (\theta - 1) \sum_i \ln x_i$$

differentiating we get

$$\ell'(\theta) = \frac{n}{\theta} + \sum_i \ln x_i$$

setting to 0 we find

$$\hat{\theta}(\underline{x}) = \frac{n}{\sum_i \ln x_i}$$

This is clearly a maximum since

$$\ell''(\theta) = -\frac{n}{\theta^2} \leq 0$$

n 10

The Likelihood function is

$$L(\theta) = \frac{1}{2^n} e^{-\sum_i |x_i - \theta|}$$

so we need to find the minimum of $\sum_i |x_i - \theta|$. Assume that the x_i are ordered with $x_i \le x_{i+1}$.

If $x_j \le \theta \le x_{j+1}$ we have

$$g(\theta) = \sum_i |x_i - \theta| = -\sum_{i=1}^{j} x_i + \sum_{i=j+1}^{n} x_i + (n-2i)\theta$$

moreover $g(\theta)$ is continuous. Thus we have 2 cases:

$n = (2m+1)$ odd: $g(\theta)$ is decreasing if $\theta < x_{m+1}$ while $g(\theta)$ is increasing if $\theta > x_{m+1}$ so that

$$\hat{\theta}(\underline{x}) = x_{m+1}$$

$n = 2m$ even: in This case $g(\theta)$ is decreasing if $\theta < x_m$ and increasing if $\theta > x_{m+1}$ while it is constant if $x_m \leq \theta \leq x_{m+1}$ so That

$\hat{\theta}(x)$ is any value in $[x_m, x_{m+1}]$

Chapter 7.6 n 9

The likelihood function is

$$L(\theta) = \frac{\beta^{n\alpha}\left(\prod x_i\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} =$$

$$\frac{\alpha^{n\alpha}}{\Gamma(\alpha)}\left(\frac{\beta}{\alpha}\right)^{n\alpha}\left(\prod_i x_i\right)^{\alpha-1} e^{-\frac{\beta}{\alpha}\alpha\sum_i x_i}$$

$$\frac{\alpha^{n\alpha}}{\Gamma(\alpha)}\gamma^{n\alpha}\left(\prod_i x_i\right)^{\alpha-1} e^{-\gamma\alpha\sum_i x_i}$$

where $\gamma = \frac{\beta}{\alpha}$ so that we need to maximize on γ. Taking the log and differentiating we get:

$$\frac{n}{\gamma} - \sum_i x_i = 0$$

or

$$\hat{\gamma}(\underline{x}) = \frac{1}{\overline{x}} .$$

7.12

Since

$$L(\theta) = \prod_{i=1}^{n} \beta e^{-\beta x_i} = \beta^n e^{-\beta n \bar{x}}$$

we get

$$\hat{\beta}(\underline{X}) = \frac{1}{\bar{X}}$$

Since $\bar{X}$ converge in probability To $\mathbb{E}(X) = \frac{1}{\beta}$. Since $f(x) = \frac{1}{x}$ is continuous for $x \neq 0$ we have That

$$\hat{\beta}(X) \xrightarrow[P]{} \frac{1}{\mathbb{E}(X)} = \beta$$

7.23

Since X_i are $B(\alpha, \beta)$ we have

$$\mathbb{E}(X_i) = \frac{\alpha}{\alpha + \beta} \qquad \mathbb{E}(X_i^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

So That

$$\bar{x} = \frac{\alpha}{\alpha+\beta} \qquad \overline{x^2} = \frac{\alpha}{\alpha+\beta} \frac{\alpha+1}{\alpha+\beta+1}$$

Solving for α and β we get

$$\hat{\alpha}_M = \overline{x}\left(\overline{x} - \overline{x^2}\right) \Big/ \left(\overline{x^2} - \overline{x}^2\right)$$

$$\hat{\beta}_M = (1 - \overline{x})\left(\overline{x} - \overline{x^2}\right) \Big/ \left(\overline{x^2} - \overline{x}^2\right)$$

For The M.L.E. we get

$$L(\theta) = \prod_i x_i^{\alpha-1} \prod_i (1-x_i)^{\beta-1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

If we Take $n=1$ we see That $\hat{\alpha}_{M.LE}$ and $\hat{\beta}_{MLE}$ are well defined while $\hat{\alpha}_M$ and $\hat{\beta}_M$ are not.