

1) In a box there are 3 red balls and 3 blue balls. You extract a ball at random. Let X be the random variable that describe the result of this extraction with $X = 1$ if the ball extracted is red and $X = 0$ if it is blue. Without reinserting the first ball you extract a second ball. Let Y be the random variable that describe the result of this second extraction with $Y = 1$ if the ball extracted is red and $Y = 0$ if it is blue.

a) find the joint p.d.f. of X and Y .

You have four possible outcome for (X, Y) : $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. It is easy to find that:

$$\begin{aligned}
 p(0, 0) &= P(X = 0 \text{ and } Y = 0) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5} \\
 p(0, 1) &= P(X = 0 \text{ and } Y = 1) = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} \\
 p(1, 0) &= P(X = 1 \text{ and } Y = 0) = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} \\
 p(1, 1) &= P(X = 1 \text{ and } Y = 1) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}
 \end{aligned}$$

b) find the marginal p.d.f of X and of Y .

$$\begin{aligned}
 p_X(0) &= p(0, 0) + p(0, 1) = \frac{1}{2} \\
 p_X(1) &= p(1, 0) + p(1, 1) = \frac{1}{2} \\
 p_Y(0) &= p(0, 0) + p(1, 0) = \frac{1}{2} \\
 p_Y(1) &= p(0, 1) + p(1, 1) = \frac{1}{2}
 \end{aligned}$$

c) find $E(X)$ and $E(Y)$.

$$\begin{aligned}
 E(X) &= 0 \cdot p_X(0) + 1 \cdot p_X(1) = \frac{1}{2} \\
 E(Y) &= 0 \cdot p_Y(0) + 1 \cdot p_Y(1) = \frac{1}{2}
 \end{aligned}$$

d) find the conditional distribution function of X given Y .

$$p_{X|Y}(0|0) = \frac{p(0,0)}{P_Y(0)} = \frac{2}{5}$$

$$p_{X|Y}(0|1) = \frac{p(0,1)}{P_Y(1)} = \frac{6}{10}$$

$$p_{X|Y}(1|0) = \frac{p(1,0)}{P_Y(0)} = \frac{6}{10}$$

$$p_{X|Y}(1|1) = \frac{p(1,1)}{P_Y(1)} = \frac{2}{5}$$

e) find $\text{Cov}(X, Y)$.

$$E(XY) = 0 \cdot 0 \cdot p(0,0) + 0 \cdot 1 \cdot p(0,1) + 1 \cdot 0 \cdot p(1,0) + 1 \cdot 1 \cdot p(1,1) = \frac{1}{5}$$

so that

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{5} - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{20}$$

f) are X and Y independent? Explain.

Clearly not because

$$p(0,0) \neq p_X(0)p_Y(0)$$

2) In a town with 1000000 voters there are two parties, party A and party B. You know that 55% of the voters prefer party A and 45% prefer party B. When next election come not all voters will vote. You know that the probability that a voter that prefer party A will vote is 0.7 and the probability that a voter that prefer party B will vote is 0.9.

a) given a randomly chosen voter, what is the probability that he/she will vote?

The probability that a random voter x will vote is

$$P(x \text{ prefer party A})P(x \text{ will vote}|x \text{ prefer party A}) + P(x \text{ prefer party B})P(x \text{ will vote}|x \text{ prefer party B}) = 0.55 \cdot 0.7 + 0.45 \cdot 0.9$$

b) given a randomly chosen voter, what is the probability that he/she will vote and vote for party A.

$$P(x \text{ will vote and vote for party A}) = P(x \text{ prefer party A})P(x \text{ will vote}|x \text{ prefer party A}) = 0.55 \cdot 0.7$$

c) given a randomly chosen voter, what is the probability that he/she will vote and vote for party B.

$$P(x \text{ will vote and vote for party B}) = P(x \text{ prefer party B})P(x \text{ will vote} | x \text{ prefer party B}) = 0.45 \cdot 0.9$$

Let S_A be the random variable describing the number of voters that will vote and vote for party A and S_B be the random variable describing the number of voters that will vote and vote for party B. Assume that every voter will decide whether to vote or not independently from the other voters.

- d) write an approximate p.d.f. for S_A , S_B and $S_A - S_B$. (**Hint:** Let X_i that r.v. that describe whether a voter that prefer party A will vote or not, *i.e.* $X_i = 1$ if he votes and $X_i = 0$ if he doesn't. Write S_A in term of the X_i .)

You have 550000 voter that prefer party A. So that

$$S_A = \sum_{i=1}^{550000} X_i$$

The X_i are Bernoulli with parameter $p = 0.7$ so that $E(X_i) = 0.7$ and $V(X_i) = 0.7 * 0.3 = 0.21$. Form the CLT we deduce that $S_A \simeq N(0.7 \cdot 550000, 0.21 \cdot 550000) = N(385000, 115500)$.
In the same way

$$S_B = \sum_{i=1}^{450000} Y_i$$

with $E(Y_i) = 0.9$ and $V(Y_i) = 0.9 * 0.1 = 0.09$ so that $S_B \simeq N(0.9 \cdot 450000, 0.09 \cdot 450000) = N(405000, 40500)$. Finally we have S_A and S_B are independent so that $S_A - S_B \simeq N(-20000, 156000)$.

- e) What is the probability that party A will win the election.

Party A wins the election if $S = S_A - S_B > 0$. We have that

$$Z = \frac{S + 20000}{\sqrt{156000}}$$

is normal standard.

$$P(S > 0) = P\left(Z > \frac{20000}{\sqrt{156000}}\right) = 1 - \Phi(50.636) \simeq 0$$

- 3) Let X de a discrete r.v. with p.m.f

$$f(i) = \begin{cases} (1 - \lambda)\lambda^i & \text{if } i \geq 0 \\ 0 & \text{if } i < 0 \end{cases}$$

Find a maximum likelihood estimator for λ . Is it unbiased? Explain.

The joint p.d.f. for a sample of size n is

$$F(i_1, i_2, \dots, i_n, \lambda) = \prod_{k=1}^n (1 - \lambda)\lambda^{i_k} = (1 - \lambda)^n \lambda^{\sum_{k=1}^n i_k}$$

so that

$$\frac{d}{d\lambda} F(i_1, i_2, \dots, i_n, \lambda) = (1 - \lambda)^n \left(\sum_{k=1}^n i_k \right) \lambda^{\sum_{k=1}^n i_k - 1} - n(1 - \lambda)^{n-1} \lambda^{\sum_{k=1}^n i_k}$$

that implies

$$\bar{i} = \frac{\lambda}{1 - \lambda} \quad \bar{i} = \frac{1}{n} \sum_{k=1}^n i_k$$

or

$$\lambda = \frac{\bar{i}}{1 + \bar{i}}$$

The estimator is biased because

$$\lambda = \frac{E(X)}{1 + E(X)} \neq E\left(\frac{X}{1 + X}\right)$$

4) Let X and Y be two r.v. with joint p.m.f given by

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ \frac{1}{2} & \text{if } -1 < x < 0 \text{ and } -1 < y < 0 \\ 0 & \text{otherwise} \end{cases}$$

a) find the marginal p.d.f. of X and Y . We have

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

If $0 < x < 1$ we have

$$f_X(x) = \frac{1}{2} \int_0^1 dy = \frac{1}{2}$$

while if $-1 < x < 0$ we have

$$f_X(x) = \frac{1}{2} \int_{-1}^0 dy = \frac{1}{2}$$

so that

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In the same way we obtain

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

b) are X and Y independent? Explain.

No because

$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{4} & \text{if } -1 < x < 1 \text{ and } -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

so that $f_X(x)f_Y(y) \neq f(x, y)$.

5) Let X be a r.v. uniformly distributed between 0 and 1. Write the p.d.f. of $Y = 2X + 1$.

Y assume value between 1 and 3 in a uniform way so that

$$f(x) = \begin{cases} 0.5 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$