A company produces personal computers and has two programs, program A and program B, to check if they work. The quality control center decides to operate as follows:

1) if the computer fails Test A (the test done using program A) then it is discarded.
2) if the computer passes Test A it is checked with Test B (the test done using program B). If it fails Test B it is discarded.
3) the computers that pass Test B are shipped to be sold.

It is observed that 95% of the computer tested with Test A pass it and 99% of those tested with Test B pass it.

After a large quality review it is found that

1) 0.1% of the shipped computer were defective.
2) 0.2% of the computer discarded because they failed Test B were working.
3) 0.2% of the computer discarded because they failed Test A were working.

Find:

a) the percentage of shipped computer on the total produced?
b) given that a computer is working what is the probability that it is discarded?
c) the probability that a shipped computer is not working?

Let first write down explicitely what the text tell us. Let A be the event { the computer passes Test A}, B the event { the computer passes Test B} and W the event { the computer is working}. Then we have:

$$P(A) = 0.95 \quad P(A') = 0.05 \quad P(B|A) = 0.99 \quad P(B'|A) = 0.01.$$

Moreover we have

$$\begin{aligned} P(W|A') &= 0.002 \quad P(W'|A') = 0.998 \\ P(W|A \cap B) &= 0.999 \quad P(W'|A \cap B) = 0.001 \\ P(W|A \cap B') &= 0.998 \quad P(W'|A \cap B') = 0.002 \end{aligned}$$

Observe that the events $A \cap B$, $A \cap B'$ and A' form a complete family of mutually disjoint events, i.e. $(A \cap B) \cup (A \cap B') \cup A' = \mathcal{S}$ and $(A \cap B) \cap (A \cap B') = \emptyset$, $(A \cap B) \cap A' = \emptyset$, $(A \cap B') \cap A' = \emptyset$.

a) *The fraction of shipped computer is:*

$$P(A \cap B) = P(A)P(B|A) = 0.95 \cdot 0.99.$$

b) *The requested probability is $P(A' \cup B'|W)$. Observe that $A' \cup B' = (A \cap B)'$. This implies that $P(A' \cup B'|W) = 1 - P(A \cap B|W)$. We have*

$$P(A \cap B|W) = P(W|A \cap B)\frac{P(A \cap B)}{P(W)}$$

and

$$\begin{aligned} P(W) =& P(W|A')P(A') + P(W|A \cap B')P(A \cap B') + P(W|A \cap B)P(A \cap B) = \\ =& P(W|A')P(A') + P(W|A \cap B')P(B'|A)P(A) + P(W|A \cap B)P(B|A)P(A). \end{aligned}$$

c) the requested probability is simply $P(W'|A \cap B) = 0.001$.

A costumer buys computer from our company. He need 5 working computer for a critical job. He decides to buy N computers with $N > 5$ because he want that the probability that at least 5 among the N he bought are working to be higher than $1 - 10^{-4}$. How large should N be?

Let X be the r.v. that counts the number of working computers among the ones he buys. Than we know that if he buys N computer we have that $X \simeq \mathrm{Bin}(0.999, N)$. The requested probability is:

$$P(X \geq 5) = \sum_{x=5}^{N} B(x; 0.999, N).$$

If $N = 5$ we have $P(X \geq 5) = (0.999)^5 = 0.995 < 1 - 10^{-4}$. So $N = 5$ is not enough. If $N = 6$ we have $P(X \geq 5) = (0.999)^6 + 6 \cdot 0.001 \cdot (0.999)^5 = 0.999985 > 1 - 10^{-4}$. So he will buy 6 computers.

Another costumer buys 3 computer every day. What is the probability that he will find the first non working computer after exactly 10 days. **Bonus** write an expression for the expected number of days he will wait before buying the first non working computer and try to evaluate it.

Let X be the r.v. that gives the day when the first non working computer is bought. The probability that the x-th computer he buys is the first non working one is given by $p(x) = 0.001 \cdot 9.999^{x-1}$. To understand observe that the probability that he buys the first non working computer the first day is $P(X = 1) = p(1) + p(2) + p(3)$, the second is $P(X = 2) = p(4) + p(5) + p(6)$, and so on. So that we get that the requested probability is:

$$P(X = 10) = p(28) + p(29) + p(30) = 0.999^{27} \cdot 0.001 \cdot (1 + 0.999 + 0.999^2)$$

The requested expected value is then:

$$\sum_{x=1}^{\infty} xP(X = x) = \sum_{x=1}^{\infty} x(1-p)p^{3(x-1)}(1 + p + p^2) = (1 - p^3) \sum_{x=1}^{\infty} x \left(p^3\right)^{(x-1)} =$$
$$=(1 - p^3) \frac{1}{(1 - p^3)^2} = \frac{1}{1 - p^3}$$

where $p = 0.999$. Do you understand this result?

The company discovers that in a group of 1000 computer ready to be shipped there are exactly 10 defective ones. Being pressed by deadlines it decides to ship the group as it is. Ten computers from this group go to one costumer. What is the probability that all five computer he bought by this custumer are working? What is the probability that four out of the five computer he bought are working?

Let X be the r.v. that count the number of working computer. Than we have that $X \simeq H(5, 1000, 990)$. We then can compute

$$P(X=5) = \frac{\binom{990}{5}\binom{10}{0}}{\binom{1000}{5}} = \frac{990 \cdot 989 \cdot 988 \cdot 987 \cdot 986}{1000 \cdot 999 \cdot 998 \cdot 997 \cdot 996}$$

and

$$P(X=4) = \frac{\binom{990}{4}\binom{10}{1}}{\binom{1000}{5}} = \frac{990 \cdot 989 \cdot 988 \cdot 987 \cdot 10 \cdot 5}{1000 \cdot 999 \cdot 998 \cdot 997 \cdot 996}$$