No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ____________________

Question:	1	2	3	4	5	6	Total
Points:	20	20	30	10	10	10	100
Score:							

Question 1 . . . . *20 point*

You play a game with a friend using a fair die with 6 faces. If the outcome is even you win \$1, if the outcome is odd you loose \$1.

(a) (10 points) Compute the probability that after 4 games you are winning exactly \$2.

Solution: If you won \$2 the only possibility is that you won 3 times and lost 1. The number of time you win is a binomial r.v. with probability of victory $p = 0.5$. Thus

$$\text{Prob. of winning \$2} = b(3; 4, 0.5) = \binom{4}{3} 2^{-4} = 0.25.$$

(b) (10 points) Use the CLT to compute the approxiamte probability that after 100 games you are winning more than \$10.

Solution: Let X_i the amount you win at the i-th game. The total amount you win is:

$$S = \sum_{i=1}^{100} X_i$$

Since $E(X_i) = 0$ and $V(X_i) = 1$ we have that S is approxiamted by a normal r.v. with mean $\mu_S = 0$ and standard deviation $\sigma_S = 10$. Thus

$$P(S > 10) \simeq P\left(\frac{S}{10} > 1\right) = \Phi(-1) = 0.159$$

Question 2 . . . . *20 point*

Let X_1 and X_2 be two independent r.v. uniformly distributed between -1 and 1. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$

(a) (10 points) Compute $\mathrm{Cov}(Y_1, Y_2)$. (**Hint**: remeber that $\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$.)

Solution: Observe that $E(X_1-X_2) = E(X_1)-E(X_2) = 0$ so that $\mathrm{Cov}(Y_1, Y_2) = E(Y_1Y_2)$. But $Y_1Y_2 = (X_1 - X_2)(X_1 + X_2) = X_1^2 - X_2^2$ so that $E(Y_1Y_2) = E(X_1^2) - E(X_2^2) = 0$ that is $\mathrm{Cov}(Y_1, Y_2) = 0$.

Alternatively, since X_1 and X_2 are independent and identically distributed, we have that

$$\mathrm{Cov}(Y_1, Y_2) = V(X_1) - V(X_2) = 0.$$

(b) (10 points) Are Y_1 and Y_2 independent? Give a mathematical argument proving your assertion. (**Hint**: if $X_1 + X_2$ is close to 2 what are the possible values of $X_1 - X_2$?)

Solution: Observe that if $X_1+X_2 > 1.8$ then both X_1 and X_2 must be greater than 0.9. This implies that $-0.2 < X_1 - X_2 < 0.2$. Thus $P(Y_1 > 1.8) \neq 0$ and $P(Y_2 > 0.2) \neq 0$ but $P(Y_1 > 1.8 \;\&\; Y_2 > 0.2) = 0$ so that Y_1 and Y_2 are not independent.

Question 3 .... *30 point*

Let T_1 and T_2 be two random variable with j.p.d.f.:

$$f(t_1, t_2) = \begin{cases} 4e^{-2t_2} & \text{if } t_1 > 0 \text{ and } t_2 > t_1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) (10 points) Compute the marginals $f_{T_1}(t_1)$ and the conditional p.d.f. $f_{T_2|T_1}(t_2|t_1)$.

Solution:

$$f_{T_1}(t_1) = \int_{t_1}^{\infty} 4e^{-2t_2} dt_2 = 2e^{-2t_1}.$$

Thus

$$f_{T_2|T_1}(t_2|t_1) = \begin{cases} 2e^{-2(t_2-t_1)} & \text{if } t_1 > 0 \text{ and } t_2 > t_1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(b) (10 points) Compute $E(T_1)$.

Solution: T_1 is an exponential r.v. of parameter 2 so that $E(T_1) = 1/2$.

(c) (10 points) Compute probability that $T_2 > 2T_1$.

Solution: We have

$$P(T_2 > 2T_1) = \int_0^\infty dt_1 \int_{2t_1}^\infty dt_2 4e^{-2t_2} = \int_0^\infty dt_1 \left(-2e^{-2t_2}\Big|_{2t_1}^\infty \right) = \int_0^\infty dt_1 2e^{-4t_1} = 0.5$$

Question 4 . . . . *10 point*

Let X be a standard normal r.v.. Let

$$Y = X^2$$

Compute the p.d.f. of Y. (**Hint**: knowing that the c.d.f. of X is $\Phi(x)$ you can compute the c.d.f of Y by writing $P(Y < y)$ in term of $P(X < x)$ for suitable values of x. Than use the realtion between c.d.f and p.d.f. Remember that the p.d.f. of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

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Solution: Oberve that

$$F_Y(y) = P(Y < y) = P(|X| < \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y}) - 1$$

so that the p.d.f. of Y is

$$f_Y(y) = \frac{d}{dy} F(y) = \frac{1}{\sqrt{y}} \Phi'(\sqrt{y}) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$$

Question 5 . . . . *10 point*

Let $X_1 \simeq N(2,9)$ and $X_2 \simeq N(2,16)$. Find that number c such that $P(|X_1 - X_2| < c) = 0.95$.

Solution:

We have that $X_1 - X_2 \simeq N(0,25)$. Thus

$$P(|X_1 - X_2| < c) = P\left(|Z| < \frac{c}{5}\right)$$

where Z is standard normal. From the tables you get that $c/5 = 1.96$ or $c = ...$

Question 6 . . . . *10 point*

In a grocery store there are two cashiers. Let N_1 be the number of people in line at the first cashier and N_2 the number of people in line at the second cashier. You know that $0 \leq N_1 \leq 2$ and $0 \leq N_2 \leq 2$. The j.p.m.f. $p(n_1, n_2)$ of N_1 and N_2 is given by:

$$p(n_1, n_2) = \begin{cases} \frac{1}{6} & \text{if } n_1 = n_2 \\ \frac{1}{8} & \text{if } n_1 = n_2 + 1 \text{ or } n_1 = n_2 - 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the $\mathrm{Cov}(N_1, N_2)$ and $\mathrm{Corr}(N_1, N_2)$.

Solution: The marginals of N_1 is

$$p_{N_1}(0) = p_{N_1}(2) = \frac{7}{24} \qquad p_{N_1}(1) = \frac{5}{12}$$

so that $E(N_1) = 1$ and $V(N_1) = 7/12$. Clearly $E(X_2) = E(X_1)$ and $V(X_2) = V(X_1)$. Finally $E(X_1X_2) = 4/3$ so that

$$\mathrm{Cov}(N_1, N_2) = \frac{4}{3} - 1^2 = \frac{1}{3} \qquad \mathrm{Corr}(N_1, N_2) = \frac{4}{7}$$