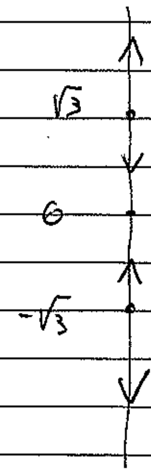
n 2

a) $x = 0$ sink

$x = \pm\sqrt{3}$ sources

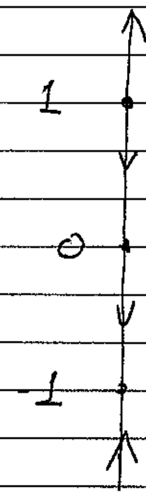


b) $\cancel{\dot{x} = x^4 - x^2}$

$x = 0$ neither sink nor source

$x = +1$ source

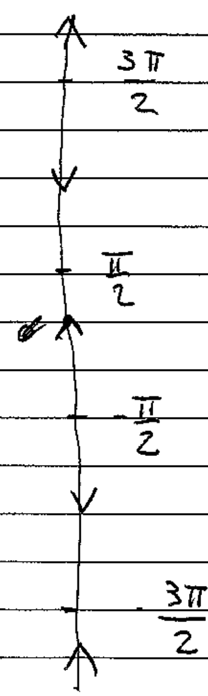
$x = -1$ sink



c) $x = (2n+1)\frac{\pi}{2}$

sink n = even

source n = odd



d) $\dot{x} = \sin^2 x$

$x = n\pi$ no sinks and no sources

e)

$x = \pm 1$ no sink

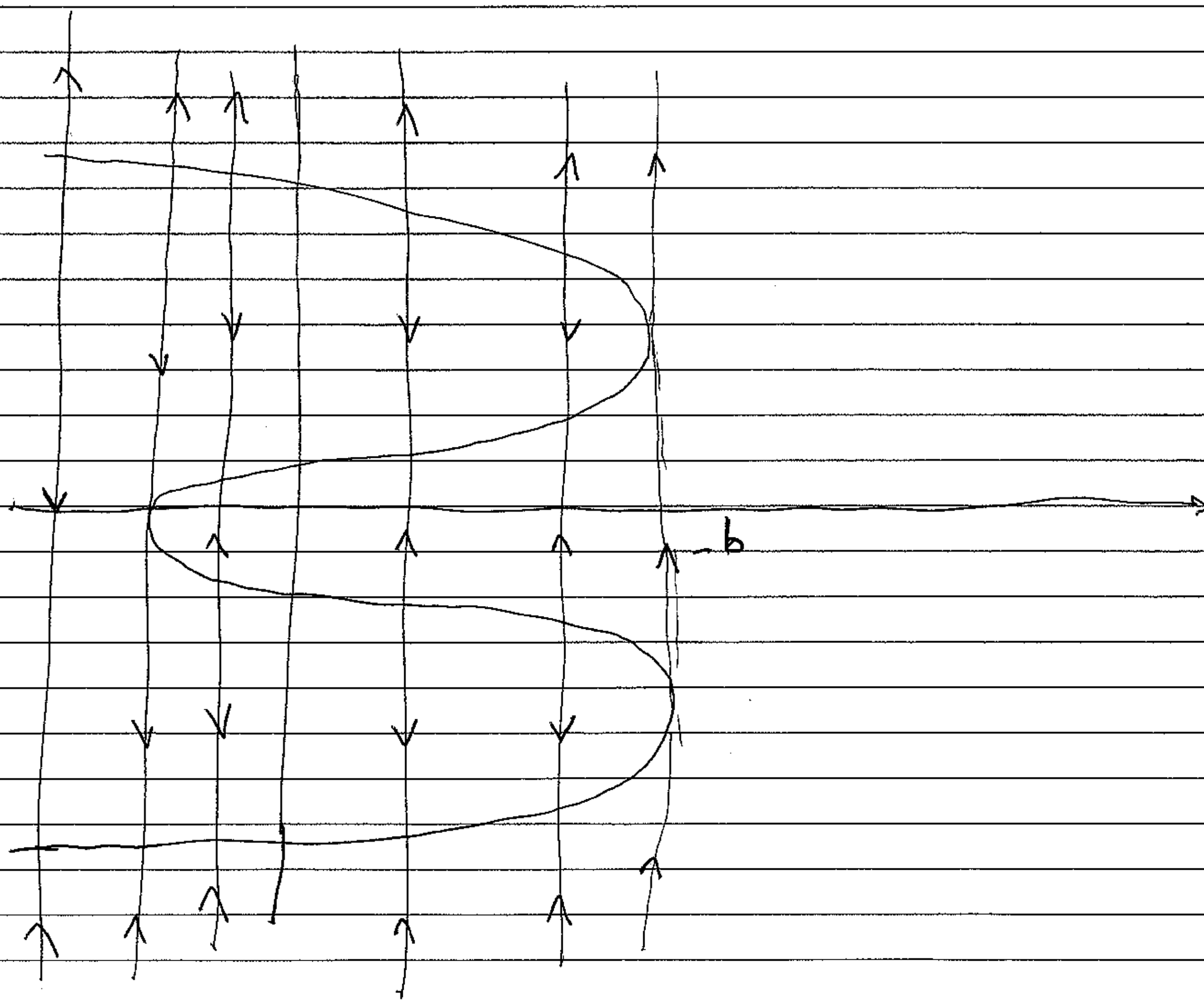
no sources

③

n 4.

a)

b)



c) let $c = f(0) > 0$ Than

* $a > -b$ no fixed point

$a = -b$ Two fixed point appear

$-b < a < c$ The Two fixed point each bifurcate

To give rise To 4 fixed point

2 sink and 2 sources

$a = c$ Two fixed point collid and disappear

$a > c$ Two fixed point, a sink and a source n 8.

Let $x(t)$ be a solution of

$$\dot{x} = ax + f(T)$$

and $y(T)$ a solution of another solution of ~~y~~ The same equation. Call

$$z(t) = x(t) - y(T)$$

it follows that $z(t)$ satisfies

$$\dot{z} = az$$

so that we Know That

$$z(t) = c e^{at}$$

is the general solution.

IT follows That

$$x(t) = c e^{at} + y(T).$$

10. It is easy to see that

$$y(t) = \frac{\sin t - \cos t}{2}$$

solves the equation. ~~Thi~~ This can be directly checked or obtained via Duhamell principle:

$$y(t) = e^{-t}\int_0^t e^s \cos(s)\,ds$$

It follows that the general solution is

$$x(t) = ce^t + \frac{\sin t - \cos t}{2} \qquad (1)$$

b) Observe that if $c \not\equiv 0$ in eq 1 then $\lim_{t\to\infty} x(t) = +\infty$ or $-\infty$ depending on the sign of c. Thus $x(t)$ is periodic only for $c = 0$ that is for $x(0) = -\frac{1}{2}$.

c) ~~Then poincare map is~~

If $x(0) = x_0$ ($p(x) =$) whe have $c - \frac{1}{2} = x_0$ or $c = \frac{1}{2} + x_0$. It follows that

$$p(x_0) = \left(\frac{1}{2} + x_0\right)e - \frac{1}{2}$$

Clearly $p(x_0) = x_0 \iff x_0 = -\frac{1}{2}$

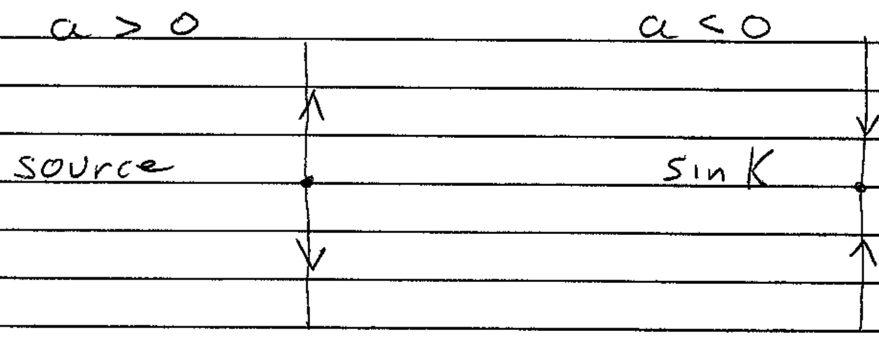
13 a) Many behaviors are possible:

I) $\dot{x} = a x^2$. In the case $a > 0$ we have:

If $x(0) > 0$ the $x(t) \to \infty$ (in a finite time) while if $x(0) < 0$ $x(t) \to 0$. In fact the solution is:

$$x(t) = \frac{x(0)}{-a\,x(0)\,t + 1}$$

II) $\dot{x} = a x^3$ The phase line in this case is



Indeed the solution is

$$x(t) = \frac{x(0)}{\left(1 - \frac{a}{2} x(0)^2 t\right)^{\frac{1}{2}}}$$

b) We can apply the same reasoning of a). If $f''(x_0)=0$ (assume $f''(0)>0$, the other case is similar). Then $f(x)>0$ $x \simeq x_0$

$\rightarrow$ if $x(0)<x_0$ we have $x(t)$ is increasing with t. We thus have that $\lim_{t\to\infty} x(t)$ exists and it can only be x_0 i.e. $\lim_{t\to\infty} x(t) = x_0$

If $x(0)>x_0$ then again $x(t)$ grows. The fact that $f''(x_0)$ allows us to say only that $x(t)$ will keep growing. We can have

$x(t) \rightarrow \infty$ like in the example in a)

$x(t) \rightarrow c$ e.g. $\dot{x} = x^2(c-x)$

~~Moreover~~

For $x(0)<x_0$ we can be more quantitative:

if $f(x)$ is smooth we can write

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + f'''(x_0)\frac{(x-x_0)^3}{6}$$

If $x(0)$ is close To x_0 Then we can neglect The Term in $(x-x_0)^3$ and we get

$$\dot{x} = \frac{f''(x_0)}{2}(x-x_0)^2$$

or calling $x - x_0 = y$ and $\frac{f''(x_0)}{2} = a$

$$\dot{y} = a y^2$$

and we can now apply The solution in a).

We have neglected The Term $(x-x_0)^3$. Is This safe? A more rigorous argument runs as follow:

if $f(x_0) = 0$ $f'(x_0) = 0$ $f''(x_0) = 2a > 0$

Then There exists $b < a$ such That, for x close To x_0 $f(x) > bx^2$ (Try To draw The picture). Suppose That I have 2 differential eq:

$$\dot{x} = f(x) \qquad x(0) = c$$

$$\dot{y} = g(y) \qquad y(0) = c$$

with $f(x) < g(x)$ for ever x.

Then

$$x(t) < g(t) \quad \text{for every } t$$

(I'll discuss this in class.)

We can thus say that, for our initial diff. eq. ~~The~~ if $x(0) < x_0$ is close to x_0 there exists $b > 0$ such that

$$x_0 > x(t) > x_0 + \frac{c}{1 - bct}$$

with $c = x(0) - x_0$.

c) Repeating the step of point b we can say that if

$$f(x_0) = 0 \quad f'(x_0) = 0 \quad f''(x_0) = 0$$

$$f'''(x_0) \neq 0$$

Then $f(x)$ changes sign when x ~~s~~ pass through x_0.

I) $f'''(x_0) > 0$. If $x(0)$ is close to x_0 then $x(t)$ grows if $x(0) > x_0$ and decreases if $x(0) < \cancel{x_0}$ ~~less~~. We can say that x_0 is a source.

Can we say anything more precise?

I) $x > x_0$ close To x_0 Than $f(x) > 0$.

We have Two possibilities:

$f(x) > 0 \ \forall x > x_0$ Then

$$\lim_{t\to\infty} x(t) \Rightarrow +\infty$$

$\exists \bar{x}$ such that $f(\bar{x}) = 0$ and

$f(x) > 0 \quad x_0 < x < \bar{x}$ then

$$\lim_{t\to\infty} x(t) = \bar{x}$$

Examples $\quad \dot{x} = x^3 \quad$ firs case

$\dot{x} = x^3(1 - x^2) \quad$ second case

II) $f'''(x_0) < 0$. Again we expect That for T large we have

$$x(t) = x_0 - \frac{c}{\left(1 + \frac{a}{2} c^2 t\right)^{\frac{1}{2}}}$$

where $c = (x_0 - x(0))$ and $a = \frac{f'''(x_0)}{6}$.

More rigorously we can say That, if $x(0)$ is close To x_0 Then There exists $b \leq \frac{f'''(x_0)}{6}$ such That

$$|x(t) - x_0| \leq \frac{|c|}{\left(1 + \frac{b}{2} c^2 t\right)^{1/2}}$$

15\.

Let $x(t)$ The solution of

$$\begin{cases} \dot{x} = f(x,t) \\ x(0) = x_0 \end{cases}$$

a) If $p < x_0 < q$ Than

$$p < x(t) < q$$

for every t. Infact assume That $\bar{t}$ is The first Time for which $x(\bar{t}) = q$, That is $x(t) < q$ for $t < \bar{t}$ and $x(\bar{t}) = q$. Since $\dot{x}(\bar{t}) = f(\bar{t}, q) < 0$ we must have $x(t) > q$ for $t < \bar{t}$ close To $\bar{t}$. But This is absurd so That $x(t) < q$ $\forall t$. The other inequality is very similar.

b) If $x_0 = q$ or $x_0 = p$ The again

$$p < x(t) < q \quad \forall t$$

(note That The inequalities are strict). Indeed if $x(0) = q$ Then $\dot{x}(0) < 0$ so That for t small $x(t) < q$. The Thesis follows from a). Again The other inequality is similar.

c) We thus have tha $p(x)$ maps the interval $[p, q]$ in an interval (p is continuous) $[p', q']$ with

$$p' \geq p \quad \text{und} \quad q' \leq q$$

so that

$$p(x) - x = \cancel{\ } \Pi(x)$$

~~rs~~ satisfies

$$\Pi(p) > 0$$

$$\Pi(q) < 0$$

Therefore there exists $\bar{x}$ such that $\Pi(\bar{x}) = 0$ or $p(\bar{x}) = \bar{x}$.

Grafically the argument is evident: