The comparison Thearem

The following is a very important statement:

Theorem: Let $X(t)$ solve

$$\begin{cases} \dot{X} = f(X) \\ X(0) = X_0 \end{cases}$$

and $Y(t)$ solve

$$\begin{cases} \dot{Y} = f(Y) \\ Y(0) = Y_0 \end{cases}$$

with

$$X_0 < Y_0$$

and

$$f(z) < g(z) \qquad \forall z$$

then

$$X(t) < Y(t) \qquad \forall t > 0$$

Proof: Suppose that for some t we have $X(t) > Y(t)$. Then there exist $\bar{t}$ such that

$$X(\bar{t}) = Y(t)$$

and $X(t) < Y(t)$ for $t < \bar{t}$

But

$$\dot{X}(\bar{t}) = f(X(\bar{t})) < g(X(\bar{t})) = g(Y(\bar{t})) = \dot{Y}(\bar{t})$$

so that

$$\frac{d}{dt}\left(X(\bar{t}) - Y(\bar{t})\right) < 0$$

This mean that $t > \bar{t}$ but close to $\bar{t}$ we have

$$X(t) < Y(t)$$

This contraddict the assumption and the Theorem is proved.

Example:

Suppose you have the equation

$$\dot{X} = f(X)$$

where $X \in \mathbb{R}^2$ and $\|f(X)\| \leq a + b\|X\|$ for every X. ~~Then~~ Moreover $f(X)$ is continuous and differentiable. Then for every X_0 there is a unique solution ~~of~~ $X(t)$ of

$$\begin{cases} \dot{X} = f(X) \\ X(0) = X_0 \end{cases}$$

with $-\infty \leq t \leq +\infty$, i.e. the equation has global solution.

Proof: Observe that

$$\frac{d}{dt}\|X\| = \frac{\dot{X}\cdot X}{\|X\|} = \frac{X\cdot f(X)}{\|X\|}$$

but

$$\frac{X\cdot f(X)}{\|X\|} \leq \frac{\|X\|\,\|f(X)\|}{\|X\|} \leq \|f(X)\| \leq$$

$$\leq \cancel{a\|X\|}\ a + b\|X\|$$

~~Let $x(t) = \|X(t)\|$ and~~

~~$g(x)$~~

Let $y(t)$ be a solution of

$$\begin{cases} \dot{y}(t) = a + b\,y(t) \\ y(0) = y_0 \end{cases}$$

with $y_0 \geq \|X_0\|$. The

$$y(t) = \left(y_0 + \frac{a}{b}\right)e^{bt} - \frac{a}{b}$$

From The Theorem it follows that

$$\|X(t)\| \leq \left(\|X_0\| + \frac{a}{b}\right)e^{bt} - \frac{a}{b}$$

Thus, from the box Theorem, $X(t)$ exists for every $t \geqslant 0$.