h 2

a) $\cancel{x}$ $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$\lambda^2 - 4\lambda + 3 = 0 \qquad \lambda_\pm = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$

$V_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad V_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$X(t) = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t}$

b) $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

$\lambda^2 - 7\lambda = 0 \qquad \lambda_+ = 7 \qquad \lambda_- = 0$

$V_+ = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad V_- = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$X(t) = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{7t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda_\pm = \frac{1 \pm 3}{2} \begin{array}{l} \nearrow \cancel{4}\ 2 \\ \searrow -1 \end{array}$$

$$V_+ = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad V_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$X(t) = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

d) $A = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}$

$$\lambda^2 + 2\lambda - 9\lambda = 0$$

$$\lambda_\pm = -1 \pm \sqrt{10}$$

n 3 (a) $\to$ 4

(b) $\to$ 2

(c) $\to$ 1

(d) $\to$ 3

6) $\ddot{x} + b\dot{x} + Kx = 0$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \dot{x} = y$

$\dot{y} = -by - Kx$

$\dot{X} = AX$

$A = \begin{pmatrix} 0 & 1 \\ -K & -b \end{pmatrix}$

Real eigenvalue if $(\mathrm{Tr}\,A)^2 - 4\det A > 0$

$b^2 - 4K^2 > 0$

If $b > 0$ and $K > 0$ This means

$b > 2K$

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4k^2}}{2}$$

$$V_{\pm} = \begin{pmatrix} 2 \\ -b \pm \sqrt{b^2 + 4k^2} \end{pmatrix}$$

$$X(t) = \alpha e^{\lambda_+ t} V_+ + \beta e^{\lambda_- t} V_-$$

Oberve That $\lambda_+ < 0$ and $\lambda_- < 0$ so That

$X(t) \to 0 \quad t \to \infty$.

7) $A = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$

Clearly for $a = 1$ A has 1 as repeated eigenvalue. In general near $a = 1$

$$\lambda^2 - (a+1)\lambda + a = 0 \qquad \lambda_{\pm} = \frac{(a+1) \pm |a-1|}{2}$$

$$\lambda_+ = a \qquad \lambda_- = 1$$

$$V_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_- = \begin{pmatrix} 1 \\ 1-a \end{pmatrix}$$

Thus $\lim_{a \to 1} V_- = V_+$. The Two eigenvector get closer and closer when $a \Rightarrow 1$.

12) ~~If~~ Observe That

$$\lambda + \mu = \mathrm{Tr}\, A \qquad \lambda \cdot \mu = \det A$$

Thus

$$\begin{pmatrix} a_{11}-\mu & a_{12} \\ a_{21} & a_{22}-\mu \end{pmatrix} \begin{pmatrix} a_{11}-\lambda \\ a_{21} \end{pmatrix} =$$

$$\begin{pmatrix} (a_{11}-\mu)(a_{11}-\lambda) + a_{12}a_{21} \\ a_{21}(a_{11}-\lambda) + a_{21}(a_{22}-\mu) \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}^2 - a_{11}\,\mathrm{Tr}\,A + \det A + a_{12}a_{21} \\ a_{12}(a_{11}+a_{22}) - a_{21}\,\mathrm{Tr}\,A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and similarly for The second line of $A - \lambda I$. So calling $V = AE_1 = \begin{pmatrix} a_{11}-\lambda \\ a_{21} \end{pmatrix}$

we have $(A - \mu I)V = 0$

and similarly for AE_2. If They are

non zero They are eigenvector for μ.

14) Let V_λ and V_μ be eigenvector.

If V_λ and V_μ are not linearly independent Then $\exists\, \alpha, \beta \neq 0$ such That

$$\alpha V_\lambda + \beta V_\mu = 0$$

it follows That

$$A(\alpha V_\lambda + \beta V_\mu) = 0$$

$$= \alpha \lambda V_\lambda + \beta \mu V_\mu$$

Let

$$B = \begin{pmatrix} \alpha & \beta \\ \alpha\lambda & \beta\mu \end{pmatrix}$$

and

$$W = \begin{pmatrix} V_{\lambda,1} \\ V_{\mu,1} \end{pmatrix}$$

we have

$$BW = 0$$

so that $\det B = \alpha\beta(\lambda - \mu) = 0$ that is impossible.

h 1

(a) $\lambda^2 - \lambda + 4 = 0$ $\lambda = \frac{+1 \pm i\sqrt{15}}{2}$

if $X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\dot{X} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ Thus

The phase portrait is 3.

(b) $\lambda^2 + \lambda + 4 = 0$ $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$

$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\dot{X} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

The phase portrait is 5

(c) $\lambda^2 - \lambda + 4 = 0$ $\lambda = \frac{1 \pm i\sqrt{15}}{2}$

phase portrait 6

(d) $\lambda^2 + 1 = 0$ $\lambda = \pm i$

$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\dot{X} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

phase portrait is 1

(e) $\lambda = \pm i$ phase portrait is 2

(f) $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$ phase portrait is 4

IV

n 2

(1) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda = 1 \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1 \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$X^*(t) = a e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Y(t) = a e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda^2 - \lambda + 1 = 0$

$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$ $\quad v_\pm = \begin{pmatrix} -2 \\ 1 \mp \sqrt{5} \end{pmatrix}$

$T = \begin{pmatrix} -2 & -2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix}$

$X(t) = a \begin{pmatrix} -2 \\ 1-\sqrt{5} \end{pmatrix} e^{(1+\sqrt{5})\frac{t}{2}} + b \begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix} e^{(1+\sqrt{5})\frac{t}{2}}$

$Y(t) = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{(1-\sqrt{5})\frac{t}{2}} + b \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{(1+\sqrt{5})\frac{t}{2}}$

(iii) $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$$\lambda^2 - \lambda + 1 = 0 \qquad \lambda_\pm = \frac{1 \pm i\sqrt{3}}{2}$$

$$V_\pm = \begin{pmatrix} -2 \\ 1 \mp i\sqrt{3} \end{pmatrix} = U \pm iW$$

$$U = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad W = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

Thus
$$T = \begin{pmatrix} -2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}$$

and
$$T^{-1} A T = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$Y(t) = a e^{\frac{t}{2}} \begin{pmatrix} \cos \frac{\sqrt{3}}{2} t \\ \sin \frac{\sqrt{3}}{2} t \end{pmatrix} + b e^{\frac{t}{2}} \begin{pmatrix} \sin \frac{\sqrt{3}}{2} t \\ -\cos \frac{\sqrt{3}}{2} t \end{pmatrix}$$

$$X(t) = e^{\frac{t}{2}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \left(a \cos \frac{\sqrt{3}}{2} t + b \sin \frac{\sqrt{3}}{2} t \right) +$$

$$e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \left(a \sin \frac{\sqrt{3}}{2} t - b \cos \frac{\sqrt{3}}{2} t \right) =$$

$$e^{\frac{t}{2}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \rho \sin\left(\frac{\sqrt{3}}{2} t + \theta \right) + e^{\frac{t}{2}} \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \rho \cos\left(\frac{\sqrt{3}}{2} t + \theta \right)$$

where $(a, b) = \rho(\cos\theta, \sin\theta)$

(c.v) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$\lambda = 2$ double

Only one eigenvector

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ Then

$$A w = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 2w + 2v_1$$

Thus $v_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ satisfy

$$A v_2 = 2 v_2 + v_1$$

$$T = (v_1 ; v_2) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$$

$$T^{-1} A T = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$Y(t) = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + b e^{2t} \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$X(t) = e^{2t} (a+b) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b e^{2t} t \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

(v) $\begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$

$\lambda_\pm = -1 \pm \sqrt{3}$ $\quad V_\pm = \begin{pmatrix} 1 \\ -2 \pm \sqrt{3} \end{pmatrix}$

$$T = \begin{pmatrix} 1 & 1 \\ -2+\sqrt{3} & -2-\sqrt{3} \end{pmatrix}$$

$$T^{-1} A T = \begin{pmatrix} -1+\sqrt{3} & 0 \\ 0 & -1-\sqrt{3} \end{pmatrix}$$

$$Y(t) = a\, e^{(-1+\sqrt{3})t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\, e^{(-1-\sqrt{3})t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X(t) = a\, e^{(-1+\sqrt{3})t} \begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} + b\, e^{(-1-\sqrt{3})t} \begin{pmatrix} 1 \\ -2-\sqrt{3} \end{pmatrix}$$

(vi) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\lambda_{\pm} = \pm\sqrt{2} \qquad V_{+} = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} \qquad V_{-} = \begin{pmatrix} \sqrt{2}-1 \\ -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & \sqrt{2}-1 \\ \sqrt{2}-1 & -1 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$$

$$Y(t) = ae^{\sqrt{2}t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + be^{-\sqrt{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

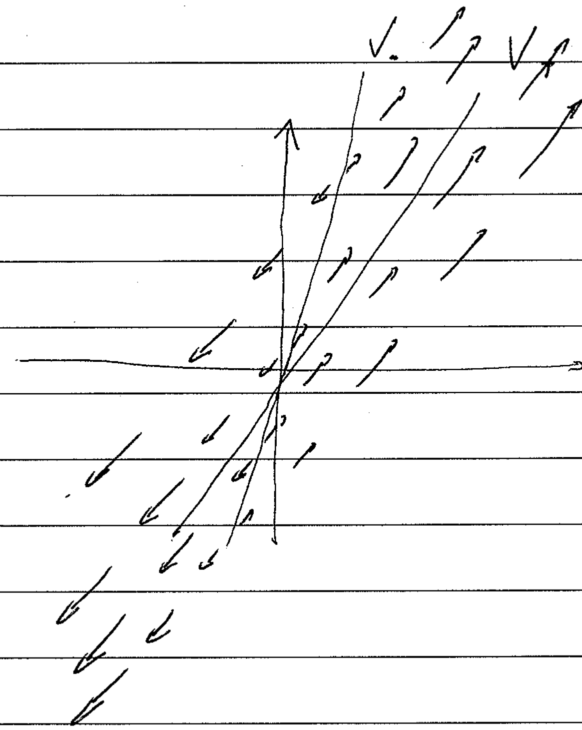
$$X(t) = ae^{\sqrt{2}t}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} + be^{-\sqrt{2}t}\begin{pmatrix} \sqrt{2}-1 \\ -1 \end{pmatrix}$$

(5) $A = \begin{pmatrix} a & 1 \\ 2a & 2 \end{pmatrix}$

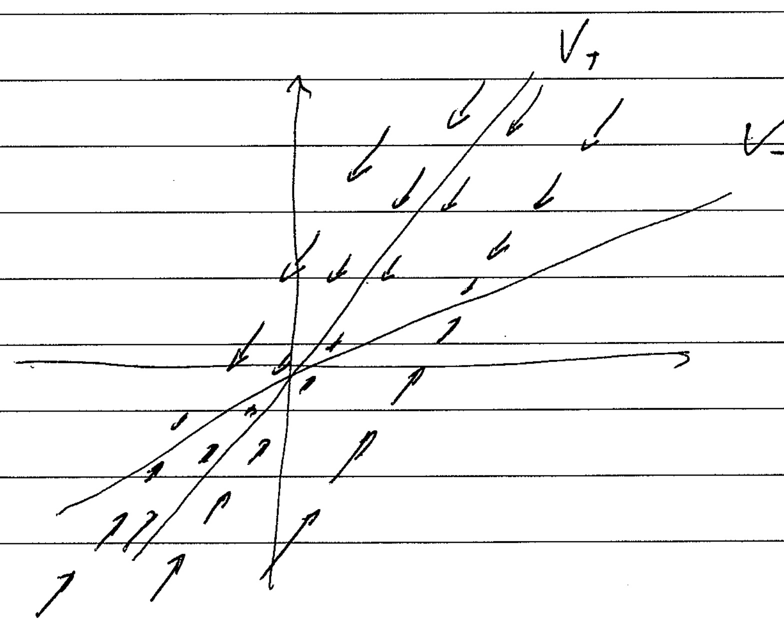
$\lambda_+ = (a+2)$ $\qquad \lambda_- = 0$

$V_+ = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\qquad V_- = \begin{pmatrix} 1 \\ -a \end{pmatrix}$

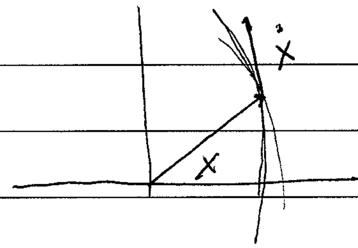
$a > -2$



$a < -2$



9) Observe That The point rotate counteclockwise if The angle θ between X and $\dot{X}$ is in $[0, \pi]$.



This means That $\sin\theta > 0$. Since

$$\sin\theta = \frac{X \times \dot{X}}{\|X\| \, \|\dot{X}\|}$$

it is enoug To ask That

$$X \times \dot{X} > 0$$

Chosing $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get

$$a_{21} > 0$$

13) ~~[illegible]~~ Exercise 2.12 Tell us that if λ is an eigenvalue than either the column of $A - \lambda I$ are eigenvector. This mean that

$$(A - \mu I)(A - \lambda I)E_i = 0$$

for $i = 1$ or 2. This mean

$$(A - \mu I)(A - \lambda I) = 0$$

The proof work even if λ and μ are complex or $\lambda = \mu$. But ~~[illegible]~~

$$\lambda + \mu = -\alpha$$
$$\lambda \mu = \beta$$

So that

$$A^2 - (\lambda + \mu)A + \lambda\mu I = 0$$
$$= A^2 + \alpha A - \beta I.$$