1) The equation governing the temperature $u(x,t)$ inside a rod is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} & 0 \leq x \leq 1 \\ \dfrac{\partial u(0,t)}{\partial x} = 0 \\ \dfrac{\partial u(1,t)}{\partial x} = r\left(T - u(1,t)\right) \\ u(x,0) = Tx \end{cases}$$

a) write and solve the equation for the steady state $v(x)$.

The steady state equation is:

$$\begin{cases} \dfrac{\partial^2 v(x)}{\partial x^2} = 0 & 0 \leq x \leq 1 \\ \dfrac{\partial v(0)}{\partial x} = 0 \\ \dfrac{\partial v(1)}{\partial x} = r\left(T - u(1)\right) \end{cases}$$

The general solution is $v(x) = ax + b$. The first boundary condition tells that $a = 0$. The second gives $T = b$.

b) write the equation for the difference $w(x,t) = u(x,t) - v(x)$.

Writing $u(x,t) = v(x) + w(x,t)$ you get the equation:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} & 0 \leq x \leq 1 \\ \dfrac{\partial w(0,t)}{\partial x} = 0 \\ \dfrac{\partial w(1,t)}{\partial x} = -ru(1,t) \\ u(x,0) = T(x-1) \end{cases}$$

c) use separation of variable to reduce the problem to a Sturm-Luiville problem. Find the eigenvalues and eigenfunctions. Explain why you can expand in eigenfunctions. Write the general solution for $w(x,t)$ and an expression for the coefficient in term of $w(x,0)$.

Writing $w(x,t) = T(t)s(x)$ we get the equations:

$$\dot{T}(t) = \mu T(t)$$

and

$$\begin{cases} \dfrac{\partial^2 s(x)}{\partial x^2} = \mu s(x) \qquad 0 \leq x \leq 1 \\ \dfrac{\partial s(0)}{\partial x} = 0 \\ \dfrac{\partial s(1)}{\partial x} = -ru(1) \end{cases}$$

The equation for s tells us that $\mu = -\lambda^2$ and

$$s(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

The first boundary condition implies that $b = 0$, so that we can take $a = 1$. The other boundary condition reads:

$$-\lambda\sin(\lambda) = -r\cos(\lambda)$$

or

$$\tan(\lambda) = \frac{r}{\lambda}$$

This equation has infinitely many solution $\lambda_n > 0$. They are the eigenvalue with eigenvector $s_n(x) = \cos(\lambda_n x)$. The general solution is then:

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos(\lambda_n x)$$

where

$$a_n = \frac{\int_0^1 \cos(\lambda_n x) w(x,0)dx}{\int_0^1 \cos^2(\lambda_n x)dx}.$$

d) write the solution of the problem. Remember that

$$\int x\cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}$$

$$\int \cos^2(\lambda x)dx = \frac{x}{2} + \frac{\sin(2\lambda x)}{4\lambda}$$

NB: there was a typo in the sign of the second term on the right in the second equation above.

$$\int_0^1 \cos^2(\lambda_n x)dx = \left(\frac{x}{2} + \frac{\sin(2\lambda_n x)}{4\lambda_n}\right)\Bigg|_0^1 = \frac{1}{2} + \frac{\sin(2\lambda_n)}{4\lambda_n} = \frac{2\lambda_n + \sin(2\lambda_n)}{4\lambda_n}$$

$$\int_0^1 \cos(\lambda_n x)w(x,0)dx = T\left(\int_0^1 x\cos(\lambda_n x)dx - \int_0^1 \cos(\lambda_n x)dx\right) = \\ = T\left(\frac{\cos(\lambda_n) - 1}{\lambda_n^2}\right)$$

so that

$$v(x,t) = T + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos(\lambda_n x)$$

with

$$a_n = \frac{4T}{\lambda_n}\frac{\cos(\lambda_n) - 1}{2\lambda_n + \sin(2\lambda_n)}$$

e) Consider the equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + k(T - u(x,t)) \qquad 0 \le x \le 1 \\ \dfrac{\partial u(0,t)}{\partial x} = 0 \\ \dfrac{\partial u(1,t)}{\partial x} = r\,(T - u(1,t)) \\ u(x,0) = Tx \end{cases}$$

Find the solution of this equation. Observe that after comnputing the steady state you can just use the results in points a), b) and c). What is the only difference with the previous case?

It is easy to check that $v(x) = T$ is still the solution of the steady state equation. Separation of variables gives the new equation

$$\frac{s(x)}{\partial x^2} - ks(x) = \mu s(x)$$

with the same boundary conditions of point c). Calling $\rho = \mu + k$ we find that there are infinitely many solution for ρ and that $\rho_n = -\lambda_n^2$ where λ_n satisfies

$$\tan(\lambda_n) = \frac{r}{\lambda_n}.$$

so that we have infinitely many μ_n with $\mu_n = -\lambda_n^2 - k$. The eigenfunction associated with μ_n is still $s_n(x) = \cos(\lambda_n x)$. Finally we have

$$v(x,t) = T + \sum_{n=1}^{\infty} a_n e^{-(\lambda_n^2 + k)t} \cos(\lambda_n x)$$

where the a_n are the one of point d). We can observe that the approach to the steady state is faster due to the presence of the term $-kt$ in the exponent.