No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	35	30	15	20	0	100
Score:						

Question:	1	2	3	4	5	Total
Bonus Points:	0	0	0	0	20	20
Score:						

Question 1 .... *35 + 0 point*

Let $f(x)$ be the periodic function of period π given by:

$$f(x) = (4x^2 - \pi^2)\cos x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

and extended periodically to all $\mathbb{R}$.

(a) (10 points) Compute $f'(x)$ and $f''(x)$.

Solution: Observe that $f(\pi/2) = f(-\pi/2)$ so that

$$f'(x) = 8x\cos x - (4x^2 - \pi^2)\sin x$$

Again we have that $f'(\pi/2) = f'(-\pi/2)$ so that

$$f''(x) = 8\cos x - 16x\sin x - (4x^2 - \pi^2)\cos x$$

(b) (10 points) Are f, f' and f'', piecewise continuous? continuous? piecewise smooth? (Justify your answer.)

Solution: We only have to check the end points. From what we said above we have that f and f' are continuous. Observe that $f''(\pi/2) = f''(-\pi/2)$ so also f'' is continuous. Finally we have

$$f'''(x) = -24\sin x - 24x\cos x - (4x^2 - \pi^2)\sin x$$

with $f'''(\pi/2) = -24$ and $f'''(-\pi/2) = 24$.
This implies that f, f' and f'' are sectionally smooth.

(c) (15 points) Compute the Fourier series for f, f' and f'' and discuss their convergence. Remember that

$$\cos a \cos b = (\cos(a+b) + \cos(a-b))/2$$

and

$$\int x^2 \cos(ax)dx = \frac{a^2x^2\sin(ax) - 2\sin(ax) + 2ax\cos(ax)}{a^3}$$

Solution: We first find the Fourier series of $cos(x)$. Since it is even we have:

$$A_0^1 = \frac{2}{\pi} \tag{1}$$

$$A_n^1 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2nx)\cos(x)\,dx = \frac{4(-1)^{n+1}}{\pi}\frac{1}{4n^2-1} \tag{2}$$

We then have to compute the Fourier series of $x^2\cos(x)$. Again we get

$$A_0^2 = -\frac{4}{\pi} + \frac{\pi}{2} \tag{3}$$

$$A_n^2 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos(2nx)\cos(x)\,dx = \tag{4}$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos((2n+1)x)\,dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos((2n-1)x)\,dx \tag{5}$$

Observe now that $\cos((2n\pm1)x) = 0$ so that the third term in the above integral does not appear. Moreover $\sin((2n+1)\pi/2) = -\sin((2n-1)\pi/2)$ so that we get

$$A_n = -\frac{2}{\pi}\left(\frac{1}{(2n+1)^3} - \frac{1}{(2n-1)^3}\right)\sin((2n+1)x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \tag{6}$$

$$+ \frac{\pi}{4}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)\sin((2n+1)x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \tag{7}$$

$$= \frac{8(-1)^n}{\pi}\frac{12n^2-1}{(4n^2-1)^3} + (-1)^{n+1}\pi\frac{1}{4n^2-1} \tag{8}$$

So that we get:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(2nx)$$

with

$$a_0 = -\frac{16}{\pi} \qquad a_n = \frac{32(-1)^n}{\pi}\frac{(12n^2-1)}{(4n^2-1)^3}$$

A Trick: From the above computation we have:

$$f^{iv}(x) = -48\cos(x) + 32x\sin(x) + (4x^2 - \pi^2)\cos(x) + 48\delta\left(x - \frac{\pi}{2}\right)$$

so that

$$2f''(x) + f^{iv}(x) + f(x) = -32\cos(x) - 48\delta\left(x - \frac{\pi}{2}\right)$$

Assuming that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

we have

$$f''(x) = -\sum_{n=0}^{\infty} 4n^2 a_n \cos(2nx) \qquad f^{iv}(x) = \sum_{n=0}^{\infty} 16n^4 a_n \cos(2nx)$$

while

$$\delta\left(x - \frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \cos(2nx)$$

so that we get, using the Fourier series of $\cos(x)$,

$$(16n^4 - 8n^2 + 1)a_n = \frac{128(-1)^n}{\pi}\frac{1}{4n^2-1} + \frac{96(-1)^n}{\pi}$$

for $n > 0$, or

$$a_0 = -\frac{16}{\pi} \qquad a_n = \frac{32(-1)^n}{\pi}\frac{(12n^2-1)}{(4n^2-1)^3}$$

Question 2 .... *30 + 0 point*

Let $f(x)$ be the function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$$

Answer the following questions.

(a) (10 points) Is f continuous?

Solution: Yes. Indeed we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and thus, due to Theorem 1, the series converge uniformly to a continuous function.

(b) (10 points) Does the Fourier series for f converge uniformly?

Solution: Yes. Indeed we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

and thus, due to Theorem 1, the series converge uniformly to a continuous function.

(c) (10 points) Is $f(x)$ sectionally smooth? (**Hint**: try to compute $f'(0)$.)

Solution: Observe that $f'(x)$, if it exists, must be given by

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)$$

so that

$$f'(0) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This implies that, if $f'(x)$ exists, it cannot be sectionally continuous so that $f(x)$ is not sectionally smooth.

Question 3 . . . . *15 + 0 point*

Consider the heat equation for a rod of length l and heat conductivity κ:

$$\begin{cases} \frac{d}{dt}u(x,t) = \kappa \frac{d^2}{dx^2}u(x,t) \\ u(0,t) = T_0 \quad u(l,t) = T_1 \\ u(x,0) = u_0(x) \end{cases}$$

If $u(x,t)$ is a solution of the above equation, set

$$x = ly \qquad\qquad t = \frac{l^2}{\kappa}s$$

and

$$v(y,s) = u\left(ly, \frac{l^2}{\kappa}s\right).$$

Write an equation for $v(y,s)$, including boundary condition and initial condition. (**Hint**: compute $dv(y,s)/ds$ and $d^2v(y,s)/dy^2$ in term of $du(x,t)/dt$ and $d^2u(x,t)/dx^2$ and use the heat equation.)

Solution: We have

$$\begin{aligned} \frac{d}{ds}v(y,s) &= \frac{d}{ds}u\left(ly, \frac{l^2}{\kappa}s\right) = \frac{l^2}{\kappa}\dot{u}(x,t) \\ \frac{d^2}{dy^2}v(y,s) &= \frac{d^2}{dy^2}u\left(ly, \frac{l^2}{\kappa}s\right) = l^2u''(x,t) \end{aligned}$$

Moreover

$$v(0,s) = u\left(0, \frac{l^2}{\kappa}s\right) = T_0$$
$$v(1,s) = u\left(l, \frac{l^2}{\kappa}s\right) = T_1$$
$$v(y,0) = u\left(ly, 0\right) = u_0(ly)$$

so that v satisfies

$$\begin{cases} \frac{d}{ds}v(y,s) = \frac{d^2}{dy^2}v(y,s) \\ v(0,s) = T_0 \quad v(1,s) = T_1 \\ v(y,0) = u_0(ly) \end{cases}$$

Question 4 .... *20 + 0 point*

Let $f(x)$ be the function:

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

(a) (10 points) Compute the Fourier transform of f. You can use real or complex notation, as you prefer.

Solution: Since f is even, we have

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)d\omega$$

where

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx = \frac{1}{2\pi}\int_{-1}^{1}\cos(\omega x)dx = \frac{1}{2\pi}\frac{\sin(\omega x)}{\omega}\bigg|_{-1}^{1} = \frac{1}{\pi}\frac{\sin(\omega)}{\omega}$$

(b) (10 points) Use the result from the previous point and the theorem of convergence of Fourier transform to compute

$$\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}\,d\omega$$

Solution: The integral to compute is

$$\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}\,d\omega = 2\int_0^\infty\frac{\sin(\omega)}{\omega}\,d\omega = \frac{\pi}{2}\int_0^\infty A(\omega)cos(\omega\cdot 0)d\omega = \frac{\pi}{2}f(0) = \frac{\pi}{2}.$$

Question 5 . . . . *0 + 20 point*

Consider the function

$$f(x) = 1 + 2\sum_{n=1}^{\infty} a^n \cos(nx)$$

with $0 < a < 1$. Find an explicit expression for $f(x)$. (**Hint:** write $\cos(nx) = (\exp(inx) + \exp(-inx))/2$ and use it to write f in complex notation. Then use that $\sum_{n=0}^{\infty} z^n = 1/(1-z)$ if $|z| < 1$.)

Solution:

We have

$$\begin{aligned} f(x) &= 1 + \sum_{n=1}^{\infty} a^n e^{inx} + \sum_{n=1}^{\infty} a^n e^{-inx} = \sum_{n=0}^{\infty} (ae^{ix})^n + \sum_{n=0}^{\infty} (ae^{-ix})^n - 1 = \\ &= \frac{1}{1 - ae^{ix}} + \frac{1}{1 - ae^{-ix}} - 1 = -\frac{1 - a^2}{1 - 2a\cos(x) + a^2} \end{aligned}$$