No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name: ______________________________

Question:	1	2	Total
Points:	70	30	100
Score:			

Question:	1	2	Total
Bonus Points:	15	10	25
Score:			

Question 1 .... *70 point*

An electric wire of lenght 1 and varying cross section $\rho(x)$ is traversed by a current I. At the left end it is insulated while at the right it is kept at constant temperature T_0 . The equation governing its temperature is thus

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \rho(x,t) I^2 \quad 0 \le x \le 1 \\ u'(0,t) = 0 \\ u(1,t) = T_0 \\ u(x,0) = T_0 \end{cases} \tag{1}$$

where T_0 is a constant.

(a) (20 points) Write the equation for the steady state $\bar{u}(x)$ of the rod. Show that a particular solution of the steady state equation is:

$$\bar{u}_p(x) = I^2 \int_0^x (y - x)\rho(y) dy\,.$$

Write the general solution and find the steady state.

Solution: The equation for the staedy state is

$$\begin{cases} \frac{d^2 u(x,t)}{dx^2} + \rho(x,t) I^2 = 0 \quad 0 \le x \le 1 \\ u'(0,t) = 0 \\ u(1,t) = T_0 \end{cases} \tag{2}$$

Observe now that

$$\frac{d}{dx}\bar{u}_p(x) = I^2 \int_0^x (y - x)\rho(y) dy = -I^2 \int_0^x \rho(y) dy$$

so that

$$\frac{d^2}{dx^2}\bar{u}_p(x) = -I^2 \frac{d}{dx} \int_0^x \rho(y) dy = -I^2 \rho(x).$$

Thus $\bar{u}_p(x)$ solves the equation so that the general solution is

$$\bar{u}(x) = a + bx + \bar{u}_p(x)$$

Observe that $u_p(0) = u_p'(0) = 0$ so that $b = 0$ while

$$a = T_0 + I^2 \int_0^1 (1 - y)\rho(y) dy\,.$$

(b) (15 points) Write the equation for the deviation $v(x,t) = u(x,t) - \bar{u}(x)$.

Solution: Clearly we get

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} & 0 \le x \le 1 \\ v'(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = \bar{T} - \bar{u}_p(x)\,. \end{cases} \tag{3}$$

where

$$\bar{T} = -I^2 \int_0^1 (1-y)\rho(y)dy$$

(c) (20 points) Use separation of variable to reduce the problem to a Sturm-Liouville problem. Find the eigenvalues and eigenfunctions.

Solution: Writing $v(x,t) = C(x)T(t)$ we get the usual equtions

$$\dot{T}(t) = -\lambda^2 T(t) \tag{4}$$

$$C''(x) = -\lambda^2 C(x) \qquad C'(0) = C(1) = 0\,. \tag{5}$$

The second equation give us

$$C(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

where $C'(0) = 0$ implyes $b = 0$ and $C(1) = 0$ inplyes

$$\cos\lambda = 0$$

that is

$$\lambda = \left(n + \frac{1}{2}\right)\pi$$

Thus eigenvalue and eigenfunction are

$$\lambda_n = \left(n + \frac{1}{2}\right)\pi \qquad C_n(x) = \cos(\lambda_n x)$$

(d) (15 points) Write the general solution of the problem with an expression for the coefficients a_n needed to match the initial condition.

Solution: We thus get that

$$u(x,t) = \bar{u}(x) + \sum_{i=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos(\lambda_n x)$$

where

$$a_n = \frac{\displaystyle\int_0^1 \cos(\lambda_n x)(\bar{T} - \bar{u}_p(x))dx}{\displaystyle\int_0^1 \cos^2(\lambda_n x)dx}$$

(e) (15 points (bonus)) Assume that

$$\rho(x) = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)\,.$$

Compute the coefficients a_n.

Solution: Observe that

$$\int_0^1 \cos^2(\lambda_n x)dx = \frac{1}{2}\,.$$

Thus we have

$$\begin{aligned}
a_n =& \frac{1}{2}\int_0^1 \cos(\lambda_n x)(\bar{T} - \bar{u}_p(x))dx = \\
=& \frac{1}{2}\frac{\sin(\lambda_n x)}{\lambda_n}(\bar{T} - \bar{u}_p(x))\Big|_0^1 + \frac{1}{2\lambda_n}\int_0^1 \sin(\lambda_n x)\bar{u}_p'(x)dx = \\
=& -\frac{1}{2}\frac{\cos(\lambda_n x)}{\lambda_n^2}\bar{u}_p'(x)\Big|_0^1 + \frac{1}{2\lambda_n^2}\int_0^1 \cos(\lambda_n x)\bar{u}_p''(x)dx = \\
=& -\frac{I^2}{2\lambda_n^2}\int_0^1 \cos(\lambda_n x)\rho(x)dx = -\frac{I^2 A_n}{2\lambda_n^2}
\end{aligned}$$

where we have used that $\bar{u}_p(1) = \bar{T}$ and $\bar{u}_p'(0) = 0$.

Question 2 .... *30 point*

Consider the Sturm-Liouville problem for $0 \leq x \leq \pi^2$:

$$\left(\sqrt{x}\phi'(x)\right)' = -\frac{\lambda^2}{4\sqrt{x}}\phi(x) \qquad \phi(0) = \phi(\pi^2) = 0\,.$$

(a) (15 points) Show that the general solution of the above differential equation is:

$$\phi(x) = a\cos(\lambda\sqrt{x}) + b\sin(\lambda\sqrt{x})\,.$$

Use the boundary conditions to find eigenvalues and eigenfunctions.

Solution: Observe that

$$\phi'(x) = -a\frac{\lambda}{2\sqrt{x}}\sin(\lambda\sqrt{x}) + b\frac{\lambda}{2\sqrt{x}}\cos(\lambda\sqrt{x})$$

so that

$$\left(\sqrt{x}\phi'(x)\right)' = -a\frac{\lambda^2}{4\sqrt{x}}\cos(\lambda\sqrt{x}) - b\frac{\lambda^2}{4\sqrt{x}}\sin(\lambda\sqrt{x}) = -\frac{\lambda^2}{4\sqrt{x}}\phi(x)\,.$$

From $\phi(0) = 0$ we get $a = 0$ while $\phi(\pi^2) = 0$ gives

$$\sin(\lambda\pi) = 0$$

so that $\lambda_n = n$ and

$$\phi_n(x) = \sin(n\sqrt{x})\,.$$

(b) (15 points) We know that we can write $\sqrt{x}$ as

$$\sqrt{x} = \sum_{i=0}^{\infty} a_n \phi_n(x) .$$

Write an expression for the coefficients a_n.

Solution: From orthogonality we know that

$$\int_0^{\pi^2} \sin(n\sqrt{x}) \sin(m\sqrt{x}) \frac{dx}{\sqrt{x}} = 0 \qquad \text{if} \qquad n \neq m$$

Thus we get

$$a_n = \frac{\displaystyle\int_0^{\pi^2} \sin(n\sqrt{x}) dx}{\displaystyle\int_0^{\pi^2} \sin^2(n\sqrt{x}) \frac{dx}{\sqrt{x}}}$$

(c) (10 points (bonus)) Compute the coefficients a_n.

Solution: By changing variable $y = \sqrt{x}$ we get

$$a_n = \frac{\displaystyle\int_0^{\pi} \sin(ny) y \, dy}{\displaystyle\int_0^{\pi} \sin^2(ny) dy} = \frac{2}{\pi} \left(y \frac{\cos(ny)}{n} \bigg|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \cos(ny) dy \right) = \frac{2(-1)^n}{n\pi}$$