MATH 4581 Test 1 June 18, 2010

No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	Total
Points:	50	10	40	100
Score:				

Question 1 .... *50 point*

Let $f(x)$ be the periodic function of period π given by:

$$f(x) = x\cos x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

and extended periodically to all $\mathbb{R}$.

(a) (10 points) Does f have any symmetry?

> **Solution:** Yes. f is odd since $f(-x) = -x\cos(-x) = -x\cos(x) = -f(x)$.

(b) (10 points) Compute $f'(x)$ and $f''(x)$.

> **Solution:**
> $$f'(x) = \cos(x) - x\sin(x)$$
> $$f''(x) = -2\sin(x) - x\cos(x)$$

(c) (10 points) Are f, f' and f'', piecewise continuous? continuous? piecewise smooth? (Justify your answer.)

> **Solution:** f is continuous since $f(\pi/2) = 0 = f(-\pi/2)$. f' is continuous since $f'(-\pi/2)) = -\pi/2 = f(-\pi/2)$. f'' has a jump discontinuity at $\pi/2$.
> So we have that
>
> - f continuous and piecewise smooth
> - f' continuous and piecewise smooth
> - f'' not continuous, picewise continuous and piecewise smooth

(d) (10 points) Compute the Fourier series for f, f' and f'' and discuss their convergence. (Remember that

$$\sin a \cos b = (\sin(a+b) + \sin(a-b))/2$$

and

$$\int x \sin(ax)dx = -\frac{x\cos(ax)}{a} + \frac{\sin(ax)}{a^2} + C.)$$

Solution: To compute the F.S. for f we just need the sine terms since the function is odd. We have:

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} x\cos x \sin(2nx)dx = \frac{2}{\pi}\int_0^{\pi/2} x\sin(2n+1)xdx + \frac{2}{\pi}\int_0^{\pi/2} x\sin(2n-1)xdx$$

Observe that $\cos(2n-1)\pi/2 = 0$ for every n,$\sin(2n-1)\pi/2 = -(-1)^n$ and $\sin(2n+1)\pi/2 = (-1)^n$ so that

$$b_n = \frac{2(-1)^n}{\pi}\left(\frac{1}{(2n+1)^2} - \frac{1}{(2n-1)^2}\right) = \frac{2(-1)^{n+1}}{\pi}\frac{8n}{(4n^2-1)^2}$$

Finally we have

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{8(-1)^{n+1}n}{(4n^2-1)^2}\sin 2nx$$

$$f'(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{16(-1)^{n+1}n^2}{(4n^2-1)^2}\cos 2nx$$

$$f''(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{32(-1)^{n}n^3}{(4n^2-1)^2}\sin 2nx$$

Clearly the F.S. for f and f' converge uniformly while the F.S. for f'' converges only pointwise.

(e) (10 points) Let $g(x)$ be the periodic function of period π given by:

$$g(x) = \sin x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

and extended periodically to all $\mathbb{R}$. Use the results of point (d) to find the Fourier series of g without doing integrals.

Solution: It is enough to observe that

$$g(x) = -\frac{f''(x) + f(x)}{2}$$

to obtain

$$g(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(32n^3 - 8n)(-1)^{n+1}}{(4n^2 - 1)^2} \sin 2nx$$

Question 2 .... *10 point*

Consider the heat equation for a rod of length l and heat conductivity κ:

$$\begin{cases} \frac{d}{dt}u(x,t) = \kappa\frac{d^2}{dx^2}u(x,t) \\ u(0,t) = T_0 \quad u(l,t) = T_1 \\ u(x,0) = u_0(x) \end{cases}$$

If $u(x,t)$ is a solution of the above equation, set

$$x = ly \qquad\qquad t = \frac{l^2}{\kappa}s$$

and

$$v(y,s) = u\left(ly, \frac{l^2}{\kappa}s\right).$$

Write an equation for $v(y,s)$, including boundary condition and initial condition. (**Hint**: compute $dv(y,s)/ds$ and $d^2v(y,s)/dy^2$ in term of $du(x,t)/dt$ and $d^2u(x,t)/dx^2$ and use the heat equation.)

Solution: Observe that

$$\frac{d}{ds}v(y,s) = \frac{l^2}{\kappa}\frac{d}{dt}u\left(ly, \frac{l^2}{\kappa}s\right) \qquad \frac{d^2}{dy^2}v(y,s) = l^2\frac{d^2}{dx^2}u\left(ly, \frac{l^2}{\kappa}s\right)$$

Substituting in the equation we have:

$$\frac{d}{ds}v(y,s) = \frac{d^2}{dy^2}v(y,s)$$

Moreover $v(0,s) = u(0,t) = T_0$ and $v(1,s) = u(l,t) = T_1$. Finally $v(y,0) = u_0(ly)$ so that the equation for v is

$$\begin{cases} \frac{d}{ds}v(y,s) = \frac{d^2}{dy^2}u(y,s) \\ v(0,s) = T_0 \quad u(1,s) = T_1 \\ v(y,0) = u_0(ly) \end{cases}$$

Question 3 .... *40 point*

Consider the boundary value problem in $[0, \pi]$:

$$\begin{cases} \dfrac{d^2}{dx^2}u(x) + (1+a)^2 u(x) = \sin 2x \\ u(0) = u(\pi) = 0 \end{cases}$$

where $a \neq 0$.

(a) (10 points) Find the solution $u_a(x)$ of the problem.

Solution: Clearly one particular solution is:

$$u_p(x) = \frac{1}{(1+a)^2 - 4} \sin 2x$$

The general solution of the homogenous is:

$$u_h(x) = b_1 \cos(1+a)x + b_2 \sin(1+a)x.$$

It is easy to see that, if $a \neq 0$, u_h cannot satisfy the boundary condition. Thus the only solution is:

$$u_a(x) = \frac{1}{(1+a)^2 - 1} \sin 2x$$

(b) (10 points) Write the solution you found in point (a) when $a = 0$. Is this the only solution of the problem for $a = 0$?

Solution: We have $u_0(x) = -\sin(2x)/3$ but it is clear that there are infinitely many solution given by:

$$u(x) = -\frac{1}{3} \sin 2x + b \sin x$$

with b generic.

(c) (10 points) Consider now the more general boundary value problem in $[0, \pi]$:

$$\begin{cases} \frac{d^2}{dx^2}u(x) + (1+a)^2 u(x) = f(x) \\ u(0) = u(\pi) = 0 \end{cases}$$

with $a \neq 0$. Write

$$f(x) = \sum_{n=1}^{\infty} f_n \sin nx$$

and

$$u(x) = \sum_{n=1}^{\infty} u_n \sin nx.$$

Find the coefficients u_n from the f_n. Use the fact that the equation is linear and has homogeneous boundary conditions.

Solution: Substituting into the equation we get:

$$\sum_{n=1}^{\infty} (-n^2 + (1+a)^2) u_n) \sin nx = \sum_{n=1}^{\infty} f_n \sin nx$$

so that for every n:

$$u_n = \frac{f_n}{-n^2 + (1+a)^2}$$

(d) (10 points) Under which conditions on f does the solution you found in point (c) admit a limit when $a \to 0$?

Solution: Clearly we need $f_1 = 0$ if not u_1 is not defined.

(e) (10 points (bonus)) Assume that f is piecewise smooth and $a \neq 0$. What can you say on the convergence of the F.S. for u? and for u'?

Solution: Since f is picewise smooth we have that f_n are bounded so that the F.S. for u converges uniformly. On the other hand we cannot say anything on the convergence of the F.S. for u'.