General Properties of Differential Equations

1.1. Initial value problem.

Let D be an open set of $\mathbb{R}\times\mathbb{R}^d$ and $f : D \to \mathbb{R}^d$, $(t, x) \mapsto f(t, x)$, be continuous. A *differential equation* is a relation

$$\dot{x}(t) = f(t, x(t)) \quad \text{or} \quad \dot{x} = f(t, x)\,, \tag{1.1}$$

where $\dot{x}(t) = \frac{d}{dt}x(t)$. We say that f is the *vector field* of the differential equation. Let $I \subset \mathbb{R}$ be an interval and let $\varphi \in C^0(I, \mathbb{R}^d)$. The function φ is said to be a *solution* of (1.1) if $(t, \varphi(t)) \in D$ for $t \in I$, $\varphi \in C^1(I, \mathbb{R}^d)$ and satisfies (1.1) for all $t \in I$. For a given $(\tau, \xi) \in D$, the *initial value problem* is to find an interval I containing τ as an interior point and a solution on I of the problem

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(\tau) &= \xi \end{aligned}\,. \tag{1.2}$$

We denote a solution of (1.2) by $x(t, \tau, \xi)$ and say that it is a solution with initial position ξ and initial time τ.

We remark that the domain D of the definition of f could be in $\mathbb{R} \times \mathbb{C}^d$ and nothing would change in the results.

Example 1.1. For the scalar initial value problem or $\mathbb{R}$, $\dot{x} = |x|^{1/2}$, $x(0) = 0$, there are at least two solutions: $\varphi = 0$, and $\psi = t^2/4$ if $t \geq 0$, $\psi = 0$, if $t \leq 0$. This example shows that, if we assume only that the function f is continuous in (1.2), then there may not be a unique solution of the initial value problem.

Example 1.2. For the initial value problem on $\mathbb{R}$, $\dot{x} = x^2$, $x(0) = 1$, the function $\varphi(t) = -(t-1)^{-1}$, is a solution which exists only on the interval $(-\infty, 1)$. Therefore, it is not always true that solutions of an initial value problem exist for all $t \in \mathbb{R}$. Can you prove that φ is the only solution of this initial value problem? Notice the difference between the smoothness properties of this vector field and the one of Example 1.1.

Theorem 1.1. *If $f \in C^0(D, \mathbb{R}^d)$, then, for any $(\tau, \xi) \in D$, there is an $\epsilon > 0$ such that the initial value problem (1.2) has a solution $x(t, \tau, \xi)$ defined on the interval $[\tau - \epsilon, \tau + \epsilon]$.*

Proof. For the given $(\tau, \xi) \in D$, we choose $\bar{\alpha} > 0$, $\bar{\beta} > 0$ such that the rectangle $R = \{(t, x) : |t - \tau| \leq \bar{\alpha},\ |x - \xi| \leq \bar{\beta}\}$ lies entirely in D. Since R is a compact set, the function f has a maximum M on R, $M = \max\{|f(t, x)| : (t, x) \in R\}$. For $0 < \alpha < \bar{\alpha}$, $0 < \beta < \bar{\beta}$, $M\alpha \leq \beta$, we let

$$\Gamma = \{\varphi \in C^0([\tau - \alpha, \tau + \alpha], \mathbb{R}^d) : \varphi(\tau) = \xi, |\varphi(t) - \xi| \leq \beta\}$$

and define the mapping $\mathcal{T} : \Gamma \to C^0([\tau - \alpha, \tau + \alpha], \mathbb{R}^d)$ by the relation

$$\mathcal{T}(\varphi)(t) = \xi + \int_\tau^t f(s, \varphi(s))ds.$$

It is easy to see that the fixed points of the mapping $\mathcal{T}$ in Γ are in one-to-one correspondence with the solutions of the initial value problem (1.2) on $[\tau - \alpha, \tau + \alpha]$. We will apply the Schauder fixed point theorem to obtain a fixed point of $\mathcal{T}$.

First of all, it is obvious that Γ is a closed, bounded convex subset of the Banach space $C^0([\tau - \alpha, \tau + \alpha], \mathbb{R}^d)$ equipped with the sup norm. For any φ in Γ, one has $\mathcal{T}(\varphi)(\tau) = \xi$ and

$$|\mathcal{T}(\varphi)(t) - \xi| \leq \left| \int_\tau^t f(s, \varphi(s))ds \right| \leq M|t - \tau| \leq M\alpha \leq \beta$$

for all t in $[\tau - \alpha, \tau + \alpha]$. As a consequence, $\mathcal{T}(\Gamma) \subset \Gamma$. Also, for any $t, s \in [\tau - \alpha, \tau + \alpha]$,

$$|\mathcal{T}(\varphi)(t) - \mathcal{T}(\varphi)(s)| \leq \left| \int_s^t f(\theta, \varphi(\theta))d\theta \right| \leq M|t - s|$$

This shows that $\mathcal{T}(\Gamma)$ is a family of equicontinuous functions. It follows from the Ascoli-Arzela theorem that the closure of $\mathcal{T}(\Gamma)$ is compact.

Since $f(t, x)$ is uniformly continuous on R, for any $\epsilon > 0$, $\varphi, \psi \in \Gamma$, there exists a $\delta > 0$ such that

$$|\mathcal{T}(\varphi)(t) - \mathcal{T}(\psi)(t)| \leq \left| \int_\tau^t (f(s, \varphi(s)) - f(s, \psi(s))ds \right| \leq \epsilon$$

for all t in $[\tau - \alpha, \tau + \alpha]$, provided that $|\varphi(t) - \psi(t)| \leq \delta$ for all $t \in [\tau - \alpha, \tau + \alpha]$. Therefore $\mathcal{T} : \Gamma \to \Gamma$ is a continuous mapping. It follows from the Schauder fixed point theorem (Corollary A.1.3) that $\mathcal{T}$ has a fixed point. This completes the proof.

Remark 1.1. Given any vector field $f \in C^0(D, \mathbb{R}^d)$, Theorem 1.1 implies that the differential equation (1.1) defines families of functions depending upon the parameters $(\tau, \xi) \in D$. If we knew that the solution of the initial value problem was unique, then this family of functions is uniquely defined. Very simple vector fields lead to familiar families of functions. For example, for the differential equation $\dot{x} = x$, with $x(0) = 1$, we have the solution e^t. For the differential equation $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$ (or, equivalently, $\ddot{x}_1 + x_1 = 0$), with $x_1(0) = 0, x_2(0) = 1$, (resp., $x_1(0) = 1, x_2(0) = -1$) we obtain the solution $\sin t$ (resp., $\cos t$). For the equation $\dot{x}_1 = x_2, \dot{x}_2 = x_1$ (or, equivalently, $\ddot{x}_1 - x_1 = 0$), with $x_1(0) = 0, x_2(0) = 1$, (respectively, $x_1(0) = 1, x_2(0) = -1$) we obtain the solution $\sinh t$ (respectively, $\cosh t$).

There are other ways to prove existence of solutions of the initial value problem which are not based on the Schauder fixed point theorem. Let us indicate one such method.

Let $\alpha > 0$ be as in the proof of Theorem 1.1. Let $\varphi : [\tau - \alpha, \tau + \alpha] \to \mathbb{R}^d$ be continuous, $\varphi(\tau) = \xi$ and $\dot{\varphi}$ be piecewise continuous. The function φ is called an ϵ-approximate solution to the initial value problem (1.2) if it satisfies

$$|\dot{\varphi}(t) - f(t, \varphi(t))| \le \epsilon$$

for all $t \in [\tau - \alpha, \tau + \alpha]$. Choose an infinite monotone sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$. For each ϵ_n, it can be shown that one can construct an ϵ_n-approximate solution φ_{ϵ_n} to the initial value problem (1.2). It follows from the Ascoli-Arzela Theorem that $\{\varphi_{\epsilon_n}\}$ has a subsequence which converges to a solution of the initial value problem (1.2). This method is called Euler's method.

Example 1.3. The solution of the initial value problem $\dot{x} = x$, $x(0) = 1$ is $x(t, 0, 1) = e^t$. We construct a solution on [0,1] by using Euler's method as follows. For each positive integer $n \in \mathbb{N}^+$, divide [0,1] into n equal intervals with length $h = 1/n$. Let $t_0, t_1, \ldots, t_n$ be the endpoints of each subinterval; that is, $t_{k+1} = h + t_k$. If $y_0 = \xi$, $y_{k+1} = y_k + hf(t_k, y_k)$, $k = 0, 1, 2, \ldots, n-1$, then we define

$$x_n(t) = y_k + \frac{t - t_k}{t_{k+1} - t_k}(y_{k+1} - y_k), \qquad \text{if } t_k \le t \le t_{k+1}.$$

It is possible to show that there is a sequence $\epsilon_n \to 0$ as $n \to \infty$ such that each $x_n(t)$ is an ϵ_n-approximate solution. We have $\lim x_n(t) = e^t$ uniformly on any compact interval as $n \to \infty$ and $x_n(1) = \left(1 + \frac{1}{n}\right)^n$, which gives another familiar definition of the value e.

Exercise 1.1. Extend the method in Example 1.3 to the general initial value problem (1.2) and show that it is possible to choose a sequence $\epsilon_n \to 0$ as $n \to \infty$ such that each $x_n(t)$ is an ϵ_n-approximate solution.

Corollary 1.2. *If $V \subset D$ is an open set whose closure belongs to D and $U \subset V$ is a compact set, then there is an $\epsilon > 0$ such that, for every $(\tau, \xi) \in U$, the initial value problem (1.2) has a solution on $(\tau - \epsilon, \tau + \epsilon)$.*

Proof. We repeat the proof of Theorem 1.1 using $\bar{\alpha}$, $\bar{\beta}$ so that the rectangle $R \subset D$ for all $(\tau, \xi) \in U$.

Remark 1.2. In the proof of the existence of a solution of the initial value problem (1.2), we used the fact that a closed bounded set in $\mathbb{R}^d$ is compact. In an infinite dimensional Banach space, the unit ball is not compact and, as a consequence, a closed bounded set may not be compact. Is it still possible to prove the existence of solutions of the initial value problem (1.2) if x is in a Banach space and f is continuous? The following nontrivial result is true.

Theorem 1.2. *If X is an infinite dimensional Banach space, then there is a continuous function f on X such that the initial value problem (1.2) has no solution in X.*

An interesting implication of this theorem is the following fact.

Theorem 1.3. *Consider the initial value problem (1.2) in a Banach space X. The space X is finite dimensional if and only if there is a solution of (1.2) for every continuous function f.*

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