2.6. Stability and Perturbations.

In this section, we give some simple results concerned with the effects of perturbations in the vector field upon the stability of a given system. They are obtained by means of the variation of constants formula and Gronwall's inequality.

Theorem 6.1. *Suppose that* $A,\, B \in C^0(\mathbb{R}, \mathbb{R}^{d\times d})$ *(or* $A,\, B \in C^0(\mathbb{R}, \mathbb{C}^{d\times d})$*) are* $d\times d$ *matrix functions and* B *satisfies*

$$L \equiv \int_0^\infty |B(s)|\, ds < \infty\,. \tag{6.1}$$

If the zero solution of (1.1) is uniformly stable on $[0,\, \infty)$*, then the zero solution of the equation*

$$\dot y = [A(t) + B(t)]y \tag{6.2}$$

is uniformly stable on $[0,\, \infty)$*.*

Proof. Let $X(t)$ is a fundamental matrix solution of (1.1) and let $y(t)$ be the solution of (6.2) with initial data $(\tau,\, \xi)$. If we consider the term $B(t)y(t)$ as a nonhomogeneous term in (6.2) and apply the variation of constants formula, then

$$y(t) = X(t)X^{-1}(\tau)\xi + \int_\tau^t X(t)X^{-1}(s)B(s)y(s)\, ds\,.$$

From our hypothesis on (1.1) and Theorem 3.1, there is a constant k such that $|X(t)X^{-1}(s)| \le k$ for $t \ge s \ge 0$. Using the formula for $y(t)$ and this estimate, we obtain

$$|y(t)| \le k|\xi| + \int_\tau^t k|B(s)||y(s)|\, ds\,.$$

An application of Gronwall's Inequality yields

$$|y(t)| \le k|\xi| e^{\int_\tau^t k|B(s)|\, ds} \le k|\xi|e^{kL}\,.$$

If $Y(t)$ is a fundamental matrix solution of (6.2), then $y(t) = Y(t)Y^{-1}(\tau)\xi$ and the above inequality shows that $|Y(t)Y^{-1}(\tau)| \le ke^{kL}$ for all $t \ge \tau \ge 0$. Theorem 3.1 implies that the zero solution of (6.1) is uniformly stable and the proof of the theorem is complete.

The proof of Theorem 6.1 is very simple and the results are not surprising to the intuition in the sense that stability should be preserved under small perturbations of the vector field. On the other hand, extreme care must be exercised on the allowable size of the perturbation. For example, the equation $\ddot u - 2t^{-1}\dot u + u = 0$ has the solutions

$\sin t - t\cos t,\ \cos t + t\sin t$ and therefore the zero solution is unstable. As $t \to \infty$, the vector field in this equation approaches the linear oscillator which is uniformly stable. The rate of approach of course is not as fast as in Theorem 6.1.

Exercise 6.1. Prove the following assertion: If A is a constant $d \times d$ matrix and e^{At} is bounded on $(-\infty, \infty)$ (that is, the zero solution of the autonomous equation $\dot{x} = Ax$ is uniformly stable on $(-\infty, \infty)$), and B satisfies (6.1), then, for any d-vector ξ, there is a unique solution $y(t)$ of $\dot{y} = [A + B(t)]y$ such that $e^{At}\xi - y(t) \to 0$ as $t \to \infty$.

Exercise 6.2. Generalize Exercise 6.1 to the case where the matrix A depends upon t.

Theorem 6.2. *Suppose that $A, B \in C(\mathbb{R}, \mathbb{R}^{d\times d})$ (or $A \in C(\mathbb{R}, \mathbb{C}^{d\times d})$) are $d \times d$ matrix functions. If the zero solution of (1.1) is uniformly asymptotically stable on $[0, \infty)$, then there is a $\delta > 0$ such that, if B satisfies*

$$|B(t)| < \delta \text{ for } t \in [0, \infty)\,, \tag{6.3}$$

the zero solution of the equation of (6.2) is uniformly asymptotically stable on $[0, \infty)$.

Exercise 6.3. Prove Theorem 6.2.

Exercise 6.4. Suppose that the matrix B in (6.2) satisfies

$$\int_{t_0}^{t} |B(s)|\, ds < \gamma(t - t_0) + c \text{ for } t \geq t_0 \geq 0\,,$$

Prove the following fact: If the zero solution of (1.1) is uniformly asymptotically stable on $[0, \infty)$, then there is a $\delta > 0$ such that, if $\gamma \leq \delta$, then the zero solution of equation (6.2) is uniformly asymptotically stable on $[0, \infty)$.

Example 6.1. It is possible to have the zero solution of (1.1) asymptotically stable (but, of course, not uniformly asymptotically stable) and to construct a perturbation B satisfying (6.1) as well as (6.3) and yet the zero solution of (6.2) is unstable. In fact, this is the case for

$$A(t) = \begin{bmatrix} -a & 0 \\ 0 & \sin\log t + \cos\log t - 2a \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} 0 & 0 \\ e^{-at} & 0 \end{bmatrix},$$

where $1 < a < 1 + e^{-\pi}$. To show that this is the case, we take the initial time as 0 and observe that the solution of (1.1) through $\xi = \text{col}\,(\xi_1, \xi_2)$ is given by

$$x_1(t) = \xi_1 e^{-at},\ x_2(t) = \xi_2 e^{t\sin\log t - 2at}$$

and the solution of (1.1) is asymptotically stable. If we choose $\xi_1 = 1$, $\xi_2 = 0$, then the solution of (6.1) is given by

$$y_1(t) = e^{-at},\ y_2(t) = e^{t \sin \log t - 2at} \int_0^t e^{-s \sin \log s}\, ds\,.$$

If we choose $\alpha \in (0, \pi/2)$ and $t_n = e^{(2n-\frac{1}{2})\pi}$, $n = 1, 2, \ldots$, then $\sin \log s \leq -\cos\alpha$ for $t_n \leq s \leq t_n e^{\alpha}$. Hence,

$$\begin{aligned}\int_0^{t_n e^{\alpha}} e^{-\sin \log s}\, ds &> \int_{t_n}^{t_n e^{\alpha}} e^{-\sin \log s}\, ds \\ &\geq \int_0^{t_n e^{\alpha}} e^{s \cos \alpha}\, ds > t_n(e^{\alpha} - 1)e^{t_n \cos \alpha}\,.\end{aligned}$$

Since $\sin \log (t_n e^{\pi}) = 1$, we have

$$|y_2(t_n e^{\pi})| \geq t_n(e^{\alpha} - 1)e^{bt_n}\,,$$

where $b = (1 - 2a)e^{\pi} + \cos \alpha$. If we choose α so that $b > 0$, then $|y_2(t_n e^{\pi})| \to \infty$ as $n \to \infty$ and the system (6.1) is unstable.

Theorem 6.3. *(Principle of Linearization) Suppose that* $f \in C^r(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, $f(t, 0) = 0$, $A(t) = \partial f(t, 0)/\partial x$ *and the function* $g(t, y) \equiv f(t, y) - A(t)y$ *satisfies the property that, for any* $\epsilon > 0$, *there is a* $\delta > 0$ *such that*

$$|g(t, y)| \leq \epsilon|y| \text{ for } |y| \leq \delta\ t \geq 0\,. \tag{6.4}$$

If the zero solution of (1.1) is uniformly asymptotically stable, then the zero solution of

$$\dot{x} = f(t, x) \tag{6.5}$$

is uniformly asymptotically stable.

Proof. If $X(t)$ is a fundamental matrix solution of (1.1), then the fact that the zero solution of (1.1) is uniformly asymptotically stable implies that there are constants $k \geq 1$, $\alpha > 0$ such that $|X(t)X^{-1}(\tau)| \leq ke^{-\alpha(t-\tau)}$ for $t \geq \tau \geq 0$. Choose $\epsilon \in (0, \alpha/k)$ and δ so that (6.4) is satisfied. Let $y(t)$ be the solution of (6.5) with initial data (τ, ξ) with $\tau \geq 0, |\xi| < (\delta/2k) < \delta$. Writing (6.5) as $\dot{y} = A(t)y + g(t, y)$ and using the variation of constants formula, we have

$$y(t) = X(t)X^{-1}(\tau)\xi + \int_{\tau}^t X(t)X^{-1}(s)g(s, y(s))\, ds\,.$$

As long as the solution $y(t)$ remains in norm $< \delta$, we can use (6.4) to obtain the estimate

$$|y(t)| \le ke^{-\alpha(t-\tau)}|\xi| + \int_\tau^t ke^{-\alpha(t-s)}\epsilon|y(s)|\,ds\,.$$

Multiplying both sides of this inequality by $e^{\alpha t}$ and applying Gronwall's inequality, we obtain

$$e^{\alpha t}|y(t)| \le ke^{\alpha\tau}|\xi|e^{\int_\tau^t k\epsilon\,ds} = ke^{\alpha\tau}|\xi|e^{k\epsilon(t-\tau)}$$

or, using the facts that $|\xi| < (\delta/2k)$, $\epsilon < \alpha/k$, we deduce that

$$|y(t)| \le k|\xi|e^{-(\alpha-k\epsilon)(t-\tau)} \le \frac{\delta}{2} \tag{6.6}$$

for all $t \ge \tau$ for which $|y(t)| < \delta$. The continuation theorem implies that $y(t)$ exists for all $t \ge \tau$ and thus the inequality (6.6) holds for all $t \ge \tau$. Since $\alpha - k\epsilon > 0$, this completes the proof of the theorem.

Corollary 6.1. *If $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \ge 1$, and there is an $x_0 \in \mathbb{R}^d$ such that $f(x_0) = 0$, and $Re\,\sigma(\partial f(x_0)/\partial x) < 0$, then the equilibrium solution x_0 of the equation $\dot{x} = f(x)$ is a local attractor.*

Exercise 6.5. Find the equilibrium points of the following equations and list the ones which are uniformly asymptotically stable:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\omega^2 \sin x_1 - 2ax_2,\ a > 0\,,$$

$$\dot{x}_1 = 2x_1 - x_1^2 - x_1x_2,\ \dot{x}_2 = -x_2 + x_1x_2\,.$$

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