2.8.7. Poincaré-Andronov-Hopf Bifurcation.

In the previous section, we have given a rather detailed method for determining the periodic orbits of a two dimensional system which is the perturbation of a linear system with a a matrix with complex purely imaginary eigenvalues. In this section, we consider the flow near the origin for the one-parameter family of differential equations

$$\dot{x} = A(\lambda)x + F(x, \lambda), \tag{8.51}$$

where $\lambda \in \mathbb{R}$, the matrix $A(\lambda)$ and function $F(x, \lambda)$ are C^k-functions with $k \geq 1$,

$$\sigma(A(0)) = \{\pm i\beta_0, \beta_0 \neq 0\}, \quad F(0, \lambda) = 0, \;\; D_x F(0, \lambda) = 0. \tag{8.52}$$

If, for $\lambda = 0$, we assume that the origin is stable (if it is unstable, we simply reverse time), then the converse theorem of Lyapunov (Theorem 5.1.1) implies the existence of a positive definite function V whose derivative is negative definite along the solutions of the equation. Therefore, if we fix a nonzero level set $V^{-1}(c)$ of V, then, for λ small, the vector field of (8.51) will point to the inside of this curve. Now, if the eigenvalues of $A(\lambda)$ have negative real parts for $\lambda < 0$ and positive real parts for $\lambda > 0$, then the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. From the Poincaré-Bendixson Theorem, it follows that there is a periodic orbit of (8.51) near the origin for $\lambda > 0$ and small; that is, there is a bifurcation of the origin into a periodic orbit. Notice that these assertions are valid even if the vector field is only continuous in λ.

The situation just described is typical. In fact, the existence of a periodic orbit near the origin can be concluded only from properties of the matrix $A(\lambda)$, the linear part of (8.51). Of course, the precise nature of the bifurcation will depend upon the nonlinear terms.

Theorem 8.4. (Poincaré–Andronov–Hopf) *Let the vector field in (8.51) be C^k with $k \geq 2$. Assume that*

$$\sigma(A(\lambda)) = \{\alpha(\lambda) \pm i\beta(\lambda)\}, \;\; \beta(0) = \beta_0 \neq 0, \alpha(0) = 0, \;\; \frac{d\alpha}{d\lambda}(0) \neq 0. \tag{8.53}$$

Then, in any neighborhood U of the origin in $\mathbb{R}^2$ and any given $\lambda_0 > 0$, there is a $\bar{\lambda}$ with $|\bar{\lambda}| < \lambda_0$ such that (8.51) has a nontrivial periodic orbit in U.

More precisely, there are constants $a_0 > 0$, $\lambda_0 > 0$, $\delta_0 > 0$, and real-valued C^1-functions $\lambda^\star(a)$, $T^\star(a)$ of a real variable a, and a $T^\star(a)$-periodic vector-valued function $\mathbf{x}^\star(t, a)$ such that, for $0 \leq a < a_0$,

(i) $\lambda^\star(0) = 0, \quad T^\star(0) = 2\pi, \quad |\mathbf{x}^\star(0, a)| = a,$

(ii) the function $\mathbf{x}^\star(t, a)$ is a solution of the system (8.51) with the parameter value $\lambda = \lambda^\star(a)$ and its components are given by

$$\begin{aligned} x_1^\star(t, a) &= a\cos t + o(|a|) \\ x_2^\star(t, a) &= -a\sin t + o(|a|) \quad \text{as } a \to 0, \end{aligned}$$

(iii) for $|\lambda| < \lambda_0$ and $|T - 2\pi| < \delta_0$, every T-periodic solution $\mathbf{x}(t)$ of (8.51) satisfying $|\mathbf{x}(0)| = a$ and $|\mathbf{x}(t)| < a_0$ must be given by the function $\mathbf{x}^\star(t, a)$, except for a possible translation in phase.

Proof. To make the subsequent computations more transparent, it is convenient to transform the linear part to a simpler form. Using a linear change of variables, we may assume that

$$A(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{pmatrix}.$$

From the assumption $\beta(0) \neq 0$, we can change the time variable so that $\beta(\lambda) = 1$ for $|\lambda|$ small. Also, the assumption $(d\alpha/d\lambda)(0) \neq 0$, in conjunction with the Inverse Function Theorem, implies that there is a one-to-one correspondence between $\alpha(\lambda)$ and λ. This permits us to use $\alpha(\lambda)$ as the parameter rather than λ. As a result of all these transformations, we may assume that the linear part of the vector field at the equilibrium point is of the form

$$A(\lambda) = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}.$$

As in the previous section, we introduce polar coordinates, $x_1 = r\cos\theta$, $x_2 = -r\sin\theta$, replace time by θ to obtain the scalar equation

$$\frac{dr}{d\theta} = \lambda r + \mathcal{P}(\lambda, r, \theta), \tag{8.54}$$

with

$$\mathcal{P}(\lambda, 0, \theta) = 0, \qquad D_r\mathcal{P}(\lambda, 0, \theta) = 0. \tag{8.55}$$

We need to investigate the 2π-periodic solutions of (8.54). In the previous subsection, we have employed the method of Lyapunov-Schmidt for these periodic solution and showed that they were in 0ne-to-one correspondence with the zeros of the bifurcation function $G(a, \lambda)$ in (8.47). The function $G(a, \lambda)$ is an odd function of a from Exercise 8.9. From (8.54), it is clear that

$$G(a, \lambda) = \lambda a + o(a) \quad \text{as } a \to 0.$$

We can now apply the Implicit Function Theorem to the function $G(a, \lambda)/a$ to conclude that, for a sufficiently small, there is a unique $\lambda^*(a)$ such that $G(a, \lambda) = 0$ if an only if $\lambda = \lambda^*(a)$. Furthermore, the function $\lambda^*(a)$ is C^1 and $\lambda^*(0) = 0$.

Now, with this $\lambda^\star(a)$, the function $r^\star(\theta, a) \equiv r(\lambda^\star(a), \theta, a)$ is a 2π-periodic solution of (8.54). As a consequence, the orbit through the point $x^0(a) = (a, 0)$ given by

$$\gamma(x^0(a)) = \{\, (x_1, x_2) \,:\, x_1 = r^\star(\theta, a)\cos\theta, \\ x_2 = -r^\star(\theta, a)\sin\theta,\ 0 \le \theta \le 2\pi \,\}$$

is a periodic orbit of the system (8.51).

To obtain the corresponding solution of (8.51), let $\theta^\star(t,\, a)$ be the solution of

$$\dot\theta = 1 + \Theta(\lambda^\star(a),\, r^\star(\theta,\, a), \theta)$$

satisfying $\theta^\star(0,\, a) = 0$. Then the minimal period of $\gamma(x^0(a))$ is determined by the first value $T^\star(a)$ for which $\theta^\star(T^\star(a),\, a) = 2\pi$. In particular, we have $T^\star(0) = 2\pi$. If we now define

$$x^\star(t,\, a) \equiv \big(r^\star(\theta^\star(t,\, a),\, a)\cos\theta^\star(t,\, a),\, -r^\star(\theta^\star(t,\, a),\, a)\sin\theta^\star(t,\, a)\big),$$

then it is not difficult to see that $x^\star$ satisfies the conditions of the theorem.

The stability type of the periodic orbits in Theorem 8.4 can be inferred from the derivative of the function $\lambda^\star(a)$ when this derivative is not zero; see Figure 8.1.

Figure 8.1. Two typical graphs of the function $\lambda^\star(a)$. Stability type of a periodic orbit $\Gamma_{\bar a}$ with amplitude $\bar a$ can be inferred from the bifurcation diagram: $\Gamma_{\bar a}$ is orbitally asymptotically stable if $d\lambda^\star(\bar a)/da > 0$, and unstable if $d\lambda^\star(\bar a)/da < 0$.

More specifically, we have the following result.

Theorem 8.5. *Let $\lambda^\star(a)$ be the function given in Theorem 8.4 and let Γ_a be the corresponding periodic orbit of (8.51). Then, for sufficiently small $a = \bar a > 0$, the periodic orbit $\Gamma_{\bar a}$ is orbitally asymptotically stable if $d\lambda^\star(\bar a)/da > 0$, and unstable if $d\lambda^\star(\bar a)/da < 0$.*

Proof. In this proof we will continue to use the notation developed in the proof of the previous theorem. We know that the stability properties of the periodic solution corresponding to the zero $(\bar a, \lambda^*(\bar a))$ of the bifurcation function is the same as the stability properties of the solution $\bar a$ of the scalar equation

$$\dot a = G(a, \lambda^*(\bar a)).$$

If sign $D_aG(\bar a, \lambda^*(\bar a)) \neq 0$, then the equilibrium point $\bar a$ is hyperbolic, being stable if this sign is > 0 and unstable if it is > 0. Since $G(a, \lambda^*(a)) = 0$ for all a sufficiently small, we have

$$D_\lambda G_\lambda(a, \lambda^*(a))\frac{d\lambda^*(a)}{da} + D_aG(a, \lambda^*(a)) = 0$$

for all a sufficiently small. From the fact that the function $G(a, \lambda) = \lambda a + o(a)$, in a sufficient small neighborhood of 0 we can assume that $D_\lambda G(a, \lambda^*(a))$ is the same sign is a if $a \neq 0$. If $\bar{a} > 0$, then we see that $\operatorname{sign} d\lambda^*(\bar{a})/da = -\operatorname{sign} D_a G(\bar{a}, \lambda^*(\bar{a}))$. This proves the theorem.

We now consider some specific examples.

Example 8.2. Consider the special planar system

$$\begin{aligned} \dot{x}_1 &= x_2 + g(\lambda,\, r^2)x_1 \\ \dot{x}_2 &= -x_1 + g(\lambda,\, r^2)x_2, \end{aligned} \tag{8.56}$$

where λ is a scalar parameter, $r^2 = x_1^2 + x_2^2$, and g satisfies $g(0, 0) = 0$ so that the origin is an isolated equilibrium point. In polar coordinates, we have

$$\dot{r} = g(\lambda,\, r^2)r, \quad \dot{\theta} = 1. \tag{8.57}$$

The bifurcation diagram for periodic solutions of (8.56) is simply a plot of the solutions of $g(\lambda,\, a^2) = 0$ in the $(\lambda,\, a)$-plane together with the λ-axis. As usual, stable periodic orbits are indicated by solid curves and unstable ones with dashed curves.

Let us now take several specific forms for g and draw the corresponding bifurcation diagrams.

- For $g(\lambda,\, r) = \lambda$: we obtain a linear perturbation of the linear harmonic oscillator. There is no nontrivial periodic orbit except at $\lambda = 0$, at which case there is one periodic orbit for each amplitude a. All periodic orbits are orbitally stable. See Figure 8.2a for the bifurcation diagram.
- For $g(\lambda,\, r) = \lambda - r^2$: there is a unique nontrivial periodic orbit if $\lambda > 0$ for a particular value of a; namely, $a = \sqrt{\lambda}$. The periodic orbit is orbitally asymptotically stable. See Figure 8.2b for the bifurcation diagram. Because the bifurcation curve emanates from the origin to the right, the bifurcation is called *supercritical*.
- For $g(\lambda,\, r) = -(r^2 - c)^2 + c^2 + \lambda$ with $c > 0$ a fixed constant: there are two nontrivial periodic orbits, one orbitally unstable and the other orbitally asymptotically stable, for $-c^2 < \lambda < 0$ with amplitudes $\left(c \pm (\lambda + c^2)^{1/2}\right)^{1/2}$. The two periodic orbits coalesce and disappear as λ decreases through $-c^2$. There is only one periodic orbit for $\lambda > 0$ and it is orbitally asymptotically stable. See Figure 8.2c for the complete bifurcation diagram. Because the bifurcation curve emanates from the origin to the left, the bifurcation is called *subcritical*.

Figure 8.2. Bifurcation diagrams of Equation (8.56): *(a)* for $g(\lambda, r^2) = \lambda$ is degenerate; *(b)* for $g(\lambda, r^2) = \lambda - r^2$ is supercritical; *(c)* for $g(\lambda, r^2) = -(r^2 - c)^2 + c^2 + \lambda$ is subcritical.

In each of these examples the hypotheses of the Poincaré–Andronov–Hopf bifurcation theorem are satisfied. The existence of a periodic orbit with small amplitude for small λ as asserted by the theorem is evident; however, the stability type of the periodic orbit depends on the nonlinear terms of the vector field. Moreover, as seen in the first and the last cases, there can also be additional periodic orbits, possibly with large amplitudes, for a given small $|\lambda|$.

Exercise 8.12. Discuss the Poincaré-Andronov-Hopf bifurcation near the origin for λ near zero of the system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2\\ \dot{x}_2 &= (\lambda - 2)x_1 + (\lambda - 1)x_2 - x_1^3 - x_1^2 x_2.\end{aligned}$$

In this example, you should first put the linear part for $\lambda = 0$ in real Jordan normal form.

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