PROOF OF THE STABLE MANIFOLD THEOREM

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Consider the system $\dot{x} = Ax + g(x)$, where $x \in \mathbb{R}^n$, A is a hyperbolic $n \times n$ matrix, g is C^1, $g(0) = 0$, and $Dg(0) = 0$.

From the definition of hyperbolic, there are complementary subspaces E^s and $E^u$of $\mathbb{R}^n$, both invariant under A, such that $A|E^s$ has only eigenvalues with negative real part, and $A|E^u$ has only eigenvalues with positive real part. There are corresponding linear projections $\Pi_s : \mathbb{R}^n \to E^s$, $\Pi_u : \mathbb{R}^n \to E^s$, such that any x equals $\Pi_s x + \Pi_u x$. (This definition implies many things, such as $\Pi_s x = x$ if and only if $x \in E^s$, $\Pi_u x = x$ if and only if $x \in E^u$, $\Pi_s x = 0$ if and only if $x \in E^u$, $\Pi_u x = 0$ if and only if $x \in E^s$.)

Π_s and Π_u commute with A. This is just another way of saying that E^s and E^u are invariant under A:

$$\Pi_s Ax = A\Pi_s x, \quad \Pi_u Ax = A\Pi_u x.$$

E^s and E^u are both invariant under e^{tA}, so Π_s and Π_u commute with A:

$$\Pi_s e^{tA} x = e^{tA}\Pi_s x, \quad \Pi_u e^{tA} x = e^{tA}\Pi_u x.$$

Recall that there exist constants $K \geq 1$ and $\alpha > 0$ such that for all $x \in \mathbb{R}^n$,

$$|e^{tA}\Pi_s x| \leq Ke^{-\alpha t}|\Pi_s x| \text{ for all } t \geq 0, \quad |e^{-tA}\Pi_u x| \leq Ke^{-\alpha t}|\Pi_u x| \text{ for all } t \geq 0.$$

Let

$$U_\epsilon = \{x \in \mathbb{R}^n : |x| \leq \epsilon\}, \quad V_\delta = \{x \in \mathbb{R}^n : |\Pi_s x| < \delta\}.$$

Recall that if 0 is an equilibrium of $\dot{x} = f(x)$ and U is a neighborhood of 0, then the *local stable set of 0 with respect to U* is

$$W^s_{\text{loc}}(0) = \{x \in U : \phi(t, x) \in U \text{ for all } t \geq 0 \text{ and } \phi(t, x) \to 0 \text{ as } t \to \infty\}.$$

Theorem 1 (Stable Manifold Theorem)**.** *Consider $\dot{x} = Ax + g(x)$ as above. For sufficiently small $\epsilon > 0$ there exists δ, $0 < \delta < \epsilon$, such that the local stable set of 0 with respect to U_ϵ, intersected with V_δ, is the graph of a C^1 function k from $\{x_s \in E^s : |x_s| < \delta$ to E^u. Moreover, $k(0) = 0$ and $Dk(0) = 0$.*

The theorem says essentially that $W^s_{\text{loc}}(0)$ is a C^1 manifold, with dimension equal to that of E^s, that is tangent to E^s at 0.

More generally, if g is C^r (respectively C^∞, respectively analytic), then k is C^r (respectively C^∞, respectively analytic).

The proof is an application of the Contraction Mapping Theorem with Parameters. But first we need a lemma.

We shall use $\mathbb{R}_+$ to mean the interval $[0, \infty)$.

Lemma 2. *Consider the inhomogeneous linear differential equation*

$$\dot{x} = Ax + h(t) \tag{1}$$

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with A as above and $h : \mathbb{R}_+ \to \mathbb{R}^n$ a bounded continuous function. Let $x_s \in E^s$. Then there is a unique function $x : \mathbb{R}_+ \to \mathbb{R}^n$ such that (i) x is a solution of (1), (ii) x is bounded, and (iii) $\Pi_s x(0) = x_s$. Moreover,

$$x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s h(s)\, ds + \int_\infty^t e^{(t-s)A}\Pi_u h(s)\, ds. \tag{2}$$

The first two terms on the right give the stable part of $x(t)$. The last term gives the unstable part.

Proof. Let $\tau \in \mathbb{R}$. Any solution of (1) can be written

$$x(t) = e^{(t-\tau)A}x(\tau) + \int_\tau^t e^{(t-s)A}h(s)\, ds. \tag{3}$$

Take $\tau = 0$ in (3) and apply Π_s to both sides. We get

$$\Pi_s x(t) = e^{tA}\Pi_s x(0) + \int_0^t e^{(t-s)A}\Pi_s h(s)\, ds.$$

In order to satisfy (iii), we have

$$\Pi_s x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s h(s)\, ds. \tag{4}$$

Now apply Π_u to both sides of (3). We get

$$\Pi_u x(t) = e^{(t-\tau)A}\Pi_u x(\tau) + \int_\tau^t e^{(t-s)A}\Pi_u h(s)\, ds. \tag{5}$$

Think of t as fixed and let $\tau \to \infty$. *If x is bounded*, then in the first summand, $t - \tau \to -\infty$ and $\Pi_u x(\tau)$ stays bounded. Therefore the first summand approaches 0, so we obtain

$$\Pi_u x(t) = \int_\infty^t e^{(t-s)A}\Pi_u h(s)\, ds. \tag{6}$$

Adding (4) and (6) we obtain (2)

What we have shown so far is uniqueness: *if* x satisfies (i), (ii), and (iii), then $x(t)$ is given by (3). To show existence, just check that (3) does in fact satisfy (i), (ii), and (iii). (i) and (iii) are pretty clear, we'll check (ii) shortly. □

To prove the Stable Manifold Theorem, define $T : C^0(\mathbb{R}_+, \mathbb{R}^n) \to C^0(\mathbb{R}_+, \mathbb{R}^n)$ by

$$Th(t) = \int_0^t e^{(t-s)A}\Pi_s h(s)\, ds + \int_\infty^t e^{(t-s)A}\Pi_u h(s)\, ds.$$

$Th(t)$ is the last two terms in (2). Actually, it's not yet clear that Th is a bounded function, but we will show that shortly.

Lemma 3. *T is a bounded linear map with $\|T\| \le \frac{2K}{\alpha}$.*

Proof. T is clearly linear. To show that T is bounded, calculate (assuming $\Pi_s = \Pi_u = 1$ for simplicity)

$$\begin{aligned}|Th(t)| &\le \int_0^t Ke^{-\alpha(t-s)}|h|\,ds + \int_t^\infty Ke^{-\alpha(s-t)}|h|\,ds\\ &= K|h|\left(e^{-\alpha t}\cdot\frac{e^{\alpha s}}{\alpha}\Big]_0^t + e^{\alpha t}\cdot\frac{e^{-\alpha s}}{-\alpha}\Big]_t^\infty\right) = K|h|\left(e^{-\alpha t}\left(\frac{e^{\alpha t}}{\alpha}-\frac{1}{\alpha}\right)+e^{\alpha t}\cdot\frac{e^{-\alpha t}}{\alpha}\right) \le \frac{2K}{\alpha}|h|.\end{aligned}$$

Therefore $|Th| \le \frac{2K}{\alpha}|h|$. This shows both that Th is a bounded function, and that T is a bounded linear map with $\|T\| \le \frac{2K}{\alpha}$. $\square$

Now we prove the Stable Manifold Theorem. Suppose $x : \mathbb{R}_+ \to \mathbb{R}^n$ is a bounded solution of $\dot{x} = Ax + g(x)$, i.e., $\dot{x} = Ax + g(x(t))$. Since $x(t)$ is bounded, so is $g(x(t))$. Hence, if $\Pi_s x(0) = x_s$, Lemma 2 tells us that

$$x(t) = e^{tA}x_s + \int_0^t e^{(t-s)A}\Pi_s g(x(s))\,ds + \int_\infty^t e^{(t-s)A}\Pi_u g(x(s))\,ds. \tag{7}$$

To express (7) more compactly, define $N : C^0(\mathbb{R}_+,\mathbb{R}^n) \to C^0(\mathbb{R}_+,\mathbb{R}^n)$ by $Nx(t) = g(x(t))$. N is a Nemytskii operator. Since g is C^1, we know that N is C^1. Then define $F : E^s \times C^0(\mathbb{R}_+,\mathbb{R}^n) \to C^0(\mathbb{R}_+,\mathbb{R}^n)$ by

$$F(x_s, x)(t) = e^{At}x_s + T\circ N(x)(t).$$

Equation (7) can now be written

$$x = F(x_s, x). \tag{8}$$

In other words, for given x_s, a solution of $\dot{x} = Ax + g(x)$ that is bounded on $\mathbb{R}_+$ and has $\Pi_s x(0) = x_s$ is a fixed point of the mapping $F(x_s,\cdot)$ from $C^0(\mathbb{R}_+,\mathbb{R}^n)$ to itself. This is the situation of the Contraction Mapping Theorem with Parameters.

Now T is bounded linear and N is C^1, so $T\circ N$ is C^1. The map $x_s \to e^{At}x_s$ from E^s to $C^0(\mathbb{R}_+,\mathbb{R}^n)$ is bounded linear. Therefore F is C^1.

Choose $\epsilon >$ small enough so that

$$\sup_{|x|\le\epsilon}\|Dg(x)\| \le \frac{\alpha}{4K}. \tag{9}$$

In this expression x is just a point in $\mathbb{R}^n$. We can do this because $Dg(0) = 0$, and $Dg(x)$ depends continuously on x because g is C^1.

Let $\delta = \frac{\epsilon}{2K}$. Since $K \ge 1$, we have $0 < \delta < \epsilon$.

Let

$$B = \{x_s \in E^s : |x_s| < \delta\}, \quad W = \{x \in C^0(\mathbb{R}_+,\mathbb{R}^n) : |x| \le \epsilon\}.$$

We claim that for each $x_s \in B$, $F(x_s,\cdot)$ (i) maps W into itself, and (ii) is a contraction of W with contraction constant $\frac{1}{2}$.

To show (i), let $x_s \in B$ and $x \in W$. Then

$$|e^{At}x_s| \le Ke^{-\alpha t}|x_s| \le K|x_s|.$$

Also, if $x \in \mathbb{R}^n$ and $|x| \le \epsilon$, then

$$|g(x)| = |g(x) - 0| = |g(x) - g(0)| \le \sup_{0\le s\le 1}\|Dg(sx)\|\,|x| \le \frac{\alpha}{4K}\epsilon.$$

Therefore if $x \in W$ then $|N(x)| \le \frac{\alpha}{4K}\epsilon$.

We conclude that if $x_s \in B$ and $x \in W$,

$$|F(x_s,x)| \le |e^{At}x_s| + |TN(x)| \le K|x_s| + \|T\|\,|N(x)| \le K\frac{\epsilon}{2K} + \frac{2K}{\alpha}\frac{\alpha}{4K}\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves (i)

To show (ii), let $x_s \in B$ and $x, y \in W$. Then

$$|g(x(t)) - g(y(t))| \le \sup_{0\le s\le 1}\left\|Dg\Big(x(t) + s\big(y(t) - x(t)\big)\Big)\right\|\,|x(t) - y(t)| \le \frac{\alpha}{4K}|x-y|.$$

Therefore $|N(x) - N(y)| \le \frac{\alpha}{4K}|x-y|$, so

$$\begin{aligned}|F(x_s,x) - F(x_s,y)| &= |TN(x) - TN(y)| = |T(N(x) - N(y))| \\ &\le \|T\|\,|N(x) - N(y)| \le \frac{2K}{\alpha}\frac{\alpha}{4K}|x-y| \le \frac{1}{2}|x-y|.\end{aligned}$$

This proves (ii).

For each $x_s \in B$ let $\psi(x_s) \in C^0(\mathbb{R}_+, \mathbb{R}^n)$ denote the fixed point of $F(x_s,\cdot)$ in W. The Contraction Mapping Theorem with Parameters says that the mapping

$$\psi : B \to W \subset C^0(\mathbb{R}_+, \mathbb{R}^n)$$

is C^1.

By construction $\Pi_s\psi(x_s)(0) = x_s$. We are interested in $\Pi_u\psi(x_s)(0)$. Define $k : E^s \to E^u$ by

$$k(x_s) = \Pi_u\psi(x_s)(0) = \Pi_u \circ \mathrm{ev}_0 \circ \psi(x_s),$$

where $\mathrm{ev}_0 : C^0(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n$ is the bounded linear map that takes the function $h(t)$ to $h(0)$. ev_0 means "evaluate at $t = 0$."

Now a fixed point of $F(0,\cdot)$ is certainly $x \equiv 0$ (it satisfies all the conditions). Since $x \equiv 0$ is in W, we must have $\psi(0) = 0$. (The first 0 is a point in E^s, the second is a constant function.) Therefore

$$k(0) = \Pi_u \circ \mathrm{ev}_0 \circ \psi(0) = 0.$$

Now

$$Dk(0) = \Pi_u \circ \mathrm{ev}_0 \circ D\psi(0).$$

To compute $D\psi(0)$, note that $\psi(x_s) = F(x_s, \psi(x_s))$ so

$$D\psi(x_s) = D_1F(x_s,\psi(x_s)) + D_2F(x_s,\psi(x_s))D\psi(x_s).$$

Taking $x_s = 0$ and applying both sides to a vector $h \in E^s$, we obtain

$$D\psi(0)h = D_1F(0,0)h + D_2F(0,0)D\psi(0)h = e^{At}h + T \circ DN(0)h = e^{At}h$$

because $DN(0) = 0$ (a consequence of $Dg(0) = 0$). Therefore

$$Dk(0)h = \Pi_u \circ \mathrm{ev}_0 \circ D\psi(0)h = \Pi_u \circ \mathrm{ev}_0(e^{At}h) = \Pi_u h = 0$$

because $h \in E^s$. Therefore $Dk(0) = 0$.

This completes the proof of the Stable Manifold Theorem.