Stable Manifold Theorem: Part 1

Lecture 31
Math 634
11/10/99

The Hartman-Grobman Theorem states that the flow generated by a smooth vector field in a neighborhood of a hyperbolic equilibrium point is topologically conjugate with the flow generated by its linearization. Hartman's counterexample shows that, in general, the conjugacy cannot be taken to be C^1. However, the Stable Manifold Theorem will tell us that there are important structures for the two flows that can be matched up by smooth changes of variable. In this lecture, we will discuss the Stable Manifold Theorem on an informal level and discuss two different approaches to proving it.

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be C^1, and let $\varphi : \mathbb{R} \times \Omega \to \Omega$ be the flow generated by the differential equation

$$\dot{x} = f(x). \tag{1}$$

Suppose that x_0 is a hyperbolic equilibrium point of (1).

Definition The (global) *stable manifold* of x_0 is the set

$$W^s(x_0) := \left\{ x \in \Omega \;\middle|\; \lim_{t\uparrow\infty} \varphi(t, x) = x_0 \right\}.$$

Definition The (global) *unstable manifold* of x_0 is the set

$$W^u(x_0) := \left\{ x \in \Omega \;\middle|\; \lim_{t\downarrow-\infty} \varphi(t, x) = x_0 \right\}.$$

Definition Given a neighborhood $\mathcal{U}$ of x_0, the local *stable manifold* of x_0 (relative to $\mathcal{U}$) is the set

$$W^s_{\text{loc}}(x_0) := \left\{ x \in \mathcal{U} \;\middle|\; \gamma^+(x) \subset \mathcal{U} \text{ and } \lim_{t\uparrow\infty} \varphi(t, x) = x_0 \right\}.$$

Definition Given a neighborhood $\mathcal{U}$ of x_0, the local *unstable manifold* of x_0 (relative to $\mathcal{U}$) is the set

$$W^u_{\text{loc}}(x_0) := \left\{ x \in \mathcal{U} \;\middle|\; \gamma^-(x) \subset \mathcal{U} \text{ and } \lim_{t\downarrow-\infty} \varphi(t, x) = x_0 \right\}.$$

Note that:

- $W^s_{\text{loc}}(x_0) \subseteq W^s(x_0)$, and $W^u_{\text{loc}}(x_0) \subseteq W^u(x_0)$.
- $W^s_{\text{loc}}(x_0)$ and $W^u_{\text{loc}}(x_0)$ are both nonempty, since they each contain x_0.
- $W^s(x_0)$ and $W^u(x_0)$ are invariant sets.
- $W^s_{\text{loc}}(x_0)$ is positively invariant, and $W^u_{\text{loc}}(x_0)$ is negatively invariant.
- $W^s_{\text{loc}}(x_0)$ is not necessarily $W^s(x_0) \cap \mathcal{U}$, and $W^u_{\text{loc}}(x_0)$ is not necessarily $W^u(x_0) \cap \mathcal{U}$.

$W^s_{\text{loc}}(x_0)$ is not necessarily invariant, since it might not be negatively invariant, and $W^u_{\text{loc}}(x_0)$ is not necessarily invariant, since it might not be positively invariant. They do, however, possess what is known as *relative invariance.*

Definition A subset $\mathcal{A}$ of a set $\mathcal{B}$ is *positively invariant relative to* $\mathcal{B}$ if for every $x \in \mathcal{A}$ and every $t \geq 0$, $\varphi(t, x) \in \mathcal{A}$ whenever $\varphi([0, t], x) \subseteq \mathcal{B}$.

Definition A subset $\mathcal{A}$ of a set $\mathcal{B}$ is *negatively invariant relative to* $\mathcal{B}$ if for every $x \in \mathcal{A}$ and every $t \leq 0$, $\varphi(t, x) \in \mathcal{A}$ whenever $\varphi([t, 0], x) \subseteq \mathcal{B}$.

Definition A subset $\mathcal{A}$ of a set $\mathcal{B}$ is *invariant relative to* $\mathcal{B}$ if it is negatively invariant relative to $\mathcal{B}$ and positively invariant relative to $\mathcal{B}$.

$W^s_{\text{loc}}(x_0)$ is negatively invariant relative to $\mathcal{U}$ and is therefore invariant relative to $\mathcal{U}$. $W^u_{\text{loc}}(x_0)$ is positively invariant relative to $\mathcal{U}$ and is therefore invariant relative to $\mathcal{U}$.

Recall that a *(k-)manifold* is a set that is locally homeomorphic to an open subset of $\mathbb{R}^k$. Although the word "manifold" appeared in the names of $W^s_{\text{loc}}(x_0)$, $W^u_{\text{loc}}(x_0)$, $W^s(x_0)$, and $W^u(x_0)$, it is not obvious from the defintions of these sets that they are, indeed, manifolds. One of the consequences of the Stable Manifold Theorem is that, if $\mathcal{U}$ is sufficiently small, $W^s_{\text{loc}}(x_0)$ and $W^u_{\text{loc}}(x_0)$ are manifolds. ($W^s(x_0)$ and $W^u(x_0)$ are what are known as *immersed* manifolds.)

For simplicity, let's now assume that $x_0 = 0$. Let $\mathcal{E}^s$ be the stable subspace of $Df(0)$, and let $\mathcal{E}^u$ be the unstable subspace of $Df(0)$. If f is linear, then $W^s(0) = \mathcal{E}^s$ and $W^u(0) = \mathcal{E}^u$. The Stable Manifold Theorem says that in the nonlinear case not only are the Stable and Unstable Manifolds indeed manifolds, but they are tangent to $\mathcal{E}^s$ and $\mathcal{E}^u$, respectively, at the origin. This is information that the Hartman-Grobman Theorem does not provide.

More precisely there are neighborhoods $\mathcal{U}^s$ of the origin in $\mathcal{E}^s$ and $\mathcal{U}^u$ of the origin in $\mathcal{E}^u$ and smooth maps $h_s : \mathcal{U}^s \to \mathcal{U}^u$ and $h_u : \mathcal{U}^u \to \mathcal{U}^s$ such that $h_s(0) = Dh_s(0) = h_u(0) = Dh_u(0) = 0$ and the local stable and unstable manifolds of 0 relative to $\mathcal{U}^s \oplus \mathcal{U}^u$ satisfy

$$W^s_{\text{loc}}(0) = \{x + h_s(x) \mid x \in \mathcal{U}^s\}$$

and

$$W^u_{\text{loc}}(0) = \{x + h_u(x) \mid x \in \mathcal{U}^u\}.$$

Furthermore, not only do solutions of (1) in the stable manifold converge to 0 as $t \uparrow \infty$, they do so exponentially quickly. (A similar statement can be made about the unstable manifold.)

Liapunov-Perron Approach

This approach to proving the Stable Manifold Theorem rewrites (1) as

$$\dot{x} = Ax + g(x), \tag{2}$$

where $A = Df(0)$. The Variation of Parameters formula gives

$$x(t_2) = e^{(t_2-t_1)A}x(t_1) + \int_{t_1}^{t_2} e^{(t_2-s)A}g(x(s))\,ds, \tag{3}$$

for every $t_1, t_2 \in \mathbb{R}$. Setting $t_1 = 0$ and $t_2 = t$, and projecting (3) onto $\mathcal{E}^s$ yields

$$x_s(t) = e^{tA_s}x_s(0) + \int_0^t e^{(t-s)A_s}g_s(x(s))\,ds,$$

where the subscript s attached to a quantity denotes the projection of that quantity onto $\mathcal{E}^s$. If we assume that the solution $x(t)$ lies on $W^s(0)$ and we set $t_2 = t$ and let $t_1 \uparrow \infty$, and project (3) onto $\mathcal{E}^u$, we get

$$x_u(t) = -\int_t^\infty e^{(t-s)A_u}g_u(x(s))\,ds.$$

Hence, solutions of (2) in $W^s(0)$ satisfy the integral equation

$$x(t) = e^{tA_s} x_s(0) + \int_0^t e^{(t-s)A_s} g_s(x(s))\, ds - \int_t^\infty e^{(t-s)A_u} g_u(x(s))\, ds.$$

Now, fix $a_s \in \mathcal{E}^s$, and define a functional T by

$$(Tx)(t) = e^{tA_s} a_s + \int_0^t e^{(t-s)A_s} g_s(x(s))\, ds - \int_t^\infty e^{(t-s)A_u} g_u(x(s))\, ds.$$

A fixed point x of this functional will solve (2), will have a range contained in the stable manifold, and will satisfy $x_s(0) = a_s$. If we set $h_s(a_s) = x_u(0)$ and define h_s similarly for other inputs, the graph of h_s will be the stable manifold.

Hadamard Approach

The Hadamard approach uses what is known as a graph transform. Here we define a functional not by an integral but by letting the graph of the input function move with the flow φ and selecting the output function to be the function whose graph is the image of the original graph after, say, 1 unit of time has elapsed.

More precisely, suppose h is a function from $\mathcal{E}^s$ to $\mathcal{E}^u$. Define its graph transform $F[h]$ to be the function whose graph is the set

$$\big\{\varphi(1, \xi + h(\xi)) \,\big|\, \xi \in \mathcal{E}^s\big\}. \tag{4}$$

(That (4) is the graph of a function from $\mathcal{E}^s$ to $\mathcal{E}^u$—if we identify $\mathcal{E}^s \times \mathcal{E}^u$ with $\mathcal{E}^s \oplus \mathcal{E}^u$—is, of course, something that needs to be shown.) Another way of putting this is that for each $\xi \in \mathcal{E}^s$,

$$F[h]((\varphi(1, \xi + h(\xi)))_s) = (\varphi(1, \xi + h(\xi)))_u;$$

in other words,

$$F[h] \circ \pi_s \circ \varphi(1, \cdot) \circ (\mathrm{id} + h) = \pi_u \circ \varphi(1, \cdot) \circ (\mathrm{id} + h),$$

where π_s and π_u are projections onto $\mathcal{E}^s$ and $\mathcal{E}^u$, respectively. A fixed point of the graph transform functional F will be an invariant manifold, and it can be show that it is, in fact, the stable manifold.