You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. All numbers appearing in the test are complex numbers and all functions are from $\mathbb{C}$ to $\mathbb{C}$.

You should solve at least one among ex. 1 and 2, one among 3 and 4, and one among 5 and 6. A total of 50 pts will give you an A.

Name: ______________________________

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	10	10	10	10	20	80
Score:								

1. (10 points) Find a one-to-one analytic function f mapping the open unit disk

$$D = \{z \mid |z| < 1\} \tag{1}$$

to the disk

$$C = \{z \mid |z - 1 - 2i| < 2\} \tag{2}$$

such that $f(0) = 1 + i$ and $f'(0)$ is real and positive. (*Hint*: you may use the result of exercise III.3.9)

Solution: We first note that the function $g(z) = 2z + 1 + 2i$ maps D into C. If $h(z)$ is a Möbius transformation that maps D into itself, $f(z) = g \circ h(z)$ maps D into C. From III.3.9 we know that

$$h(z) = \frac{az + b}{\bar{b}z + \bar{a}} \tag{3}$$

We thus have that

$$f(z) = \frac{2az + 2b}{\bar{b}z + \bar{a}} + 1 + 2i \tag{4}$$

We get

$$\begin{aligned} f(0) &= \frac{2b}{\bar{a}} + 1 + 2i && \Rightarrow && \frac{2b}{\bar{a}} = -i \\ f'(0) &= \frac{2|a|^2 - 2|b|^2}{\bar{a}^2} && \Rightarrow && \frac{|a|^2 - |b|^2}{\bar{a}^2} > 0 \end{aligned} \tag{5}$$

The second line implies that a is either real or pure immaginary. We can thus reduce the choice to $a = 1$ or $a = i$. But the first line tells that $2|b| = |a|$ thus $a = 1$ and $b = -i/2$.

2. (10 points) Construct a Möbius transformation that maps the circle $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ onto two concentric circles. (*Hint*: find two points z and z^* that are conjugate with respect to both circles and ...)

> **Solution:** By symmetry we have that if z and z^* are conjugate with respect to both circles they have to be real. This implies that they satisfy the equation:
>
> $$\left(z - \frac{1}{4}\right)\left(\frac{1}{z} - \frac{1}{4}\right) = \frac{1}{16} \tag{6}$$
>
> which admits as solutions $z = 2 - \sqrt{3}$ and $z* = 2 + \sqrt{3}$. Given a circle Γ, its center is always conjugated with ∞. If T is a Möbius transformation such that $T(z^*) = \infty$, then $T(z)$ is the point conjugated to ∞ with respect to the images under T of both circles. Thus $T(z)$ is the center of the images under T of both circles. Thus T maps the two circles to two concentric circles. Clearly we can also use the maps S such that $S(z) = \infty$.

3. (10 points) Let f be an entire function. Given n, suppose that:

$$\sup_{|z|>1} \left| \frac{f(z)}{z^n} \right| < C \tag{7}$$

for some constant C. Give a characterization of f.

Solution: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Call $p(z) = \sum_{k=0}^{n-1} a_k z^k$. Finally call

$$g(z) = \frac{f(z) - p(z)}{z^n} \tag{8}$$

Clearly $g(z)$ is entire since it is well defined at $z = 0$. Thus $g(z)$ is bounded on $|z| \leq 1$ (since it is continuous) and on $|z| > 1$ (this follows immediately from the hypothesis). Thus g is a constant and f is a polynomial of degree n.

4. (10 points) Let f_n be analytic functions in the open unit disk $D = \{z \,|\, |z| < 1\}$ that converge to a function f. Assume that f_n converge uniformly to f in any closed disk $D_r = \{z \,|\, |z| \leq r\}$ of radius $r < 1$. Prove that f is analytic on the open unit disk. (*Hint*: given two analytic functions f and g, estimate $|f'(z) - g'(z)|$ in terms of $|f(w) - g(w)|$)

Solution: First solution
Let $r < 1$ and $\gamma = \{re^{2is\pi} | 0 \leq s \leq 1\}$. Since f_n is analytic we have

$$f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{(w-z)^2} dw \tag{9}$$

for every $|z| < r$. This implies that $|f_n'(z) - f_m'(z)| \leq \sup_{|z|=r} |f_n(z) - f_m(z)|(r - |z|)^{-2}$. Since f_n converge uniformly and r is arbitrary, we get that $f_n'(z)$ is a Cauchy sequence on every D_r with $r < 1$. Thus it converge uniformly to a function g on every D_r. It is now enough to observe that if $f_n \to f$ uniformly and $f_n' \to g$ uniformely, than $f' = g$. In our case it is enough to observe that for every $z, z' \in D_r$ we have:

$$|f_n(z) - f_n(z') - f_n'(z)(z - z')| \leq C|z - z'|^2 \tag{10}$$

where $C = \sup_n \sup_{D_r} f_n''(z)$ exists due to an argument similar to the above one. Taking the limit we get:

$$|f(z) - f(z') - g(z)(z - z')| \leq C|z - z'|^2 \tag{11}$$

that means that g is the derivative of f.

Second solution
Let $r < 1$ and $\gamma = \{re^{2is\pi} | 0 \leq s \leq 1\}$. Since f_n is analytic we have

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w-z} dw \tag{12}$$

for every $|z| < r$. Taking the limit on n on both side we get

$$f(z) = \int_\gamma \frac{f(w)}{w-z} dw \tag{13}$$

From Proposition IV.2.1 we get that $f(z)$ is differentiable with respect to both $\Re(z)$ and $\Im(z)$. Since $f(w)/(w-z)$ is an analytic function of z for every w we also get that the derivatives of f satisfies the Cauchy-Riemann equations and thus f is analytic for $|z| < r$. Since r was arbitrary we obtain the thesis.

5. (10 points) Let f_n be analytic functions in the closed unit disk $D = \{z \,|\, |z| \le 1\}$. Suppose that f_n converge uniformely to an analityc function f on D. Let, moreover, $a_k^{(n)}$ be the coefficients of the Taylor series of f_n around 0, *i.e.*

$$f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} x^k \tag{14}$$

Similarly let a_k be the coefficients of the Taylor series of f around 0, *i.e.*

$$f(z) = \sum_{k=0}^{\infty} a_k x^k \tag{15}$$

Define $\epsilon_n = \sup_k |a_k^{(n)} - a_k|$.

(a) Show that $\lim_{n\to\infty} \epsilon_n = 0$.

Solution: Let $\gamma = \{e^{2\pi i s} | 0 \le s \le 1\}$. Since we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(w)}{(w-z)^{k+1}} dw \tag{16}$$

we get $|f_n^{(k)}(0) - f^{(k)}(0)| \le k!\epsilon$ if n is so large that $|f_n(z) - f(z)| \le \epsilon$. The thesis follows immediately from the fact that $a_k^{(n)} = f_n^{(k)}(0)/k!$.

(b) Is the converse true? More precisely, is it true that if $\lim_{n\to\infty} \epsilon_n = 0$ then f_n converges uniformly to f.

Solution: Let $a_k^{(n)} = 1/n$ if $k < n$ and $a_k^{(n)} = 0$ if $k > n$. Clearly $\epsilon_n = 1/n$ and $\lim_{n\to\infty} \epsilon_n = 0$. On the other hand, $f_n(1) = 1$ so that f_n does not converge uniformely to 0.

6. (10 points) Show that there exists a function f with $f(0) = 0$ that satisfies the differential equation

$$f'(z) = z \exp(f(z)) \tag{17}$$

in a region containing the origin. Show that the function f is analitic in a disc centered in $z = 0$ and compute the power series representation of f around $z = 0$. Find the radius of convergence of this series.

Solution: Observe that the equation can be rewritten as

$$\frac{d}{dz} \exp(-f(z)) = -z \tag{18}$$

or

$$\exp(-f(z)) = 1 - \frac{z^2}{2} \tag{19}$$

where we have used that $f(0) = 1$. Since $1 - \frac{z^2}{2} \neq 0$ for $|z| < \sqrt{2}$ we have that

$$f(z) = -\ln\left(1 - \frac{z^2}{2}\right) \tag{20}$$

This implies that $f(z)$ as a power series expansion around $z = 0$ that can be immediately obtained from the expansion of $\ln(1 - z)$. We get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{21}$$

where $a_n = 0$ if n is odd and

$$a_{2m} = \frac{1}{m} \frac{z^n}{2^m} \tag{22}$$

if $n = 2m$ is even. Moreover, since there is no singularity in $|z| < \sqrt{2}$ we clearly have that the convergence radius is $\sqrt{2}$.

7. (20 points) Let f be a continuous function defined in a connected region G of $\mathbb{C}$. Suppose that
$$\int_\gamma f(w)dw = 0 \tag{23}$$
for every closed smooth curve $\gamma \in G$. Prove that f is analytic. (*Hint*: show first that f has a primitive.)

Solution: Fix a point $a \in G$. For every $z \in G$ and every smooth curve γ connecting a to z define:
$$F(z,\gamma) = \int_\gamma f(w)dw \tag{24}$$
Since G is connected there exists at least one such γ. If γ' and γ'' both connect a to z then $\gamma = \gamma' + (-\gamma'')$ is a closed curve. Thus
$$0 = \int_\gamma f(w)dw = \int_{\gamma'} f(w)dw - \int_{\gamma''} f(w)dw = F(z,\gamma') - F(z,\gamma'') \tag{25}$$
so that $F(z,\gamma) \equiv F(z)$ does not depend on γ.

Clearly we have
$$F(z') - F(z) = \int_\sigma f(w)dw \tag{26}$$
where σ is any curve connecting z to z'. For every $z \in G$ there is d such that $B(z,d) \subset G$. If $z' \in B(z,d)$, we can choose $\sigma(t) = (1-t)z + tz'$. We get
$$\frac{F(z') - F(z)}{z - z'} - f(z) = \int_\sigma \left(f((1-t)z + tz') - f(z)\right) dw \tag{27}$$
Since f is continous, for every ϵ we can find δ such that $|f((1-t)z + tz') - f(z)| < \epsilon$ if $|z - z'| < \delta$. Thus we get that $F'(z) = f(z)$ for every $z \in G$. It is clear now that F is analytic since it admits a continuous derivative. It follows that $F''(z) = f'(z)$ exists and is continuous for every $z \in G$. This proves that f is analytic in G.